HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Matemaattis-luonnontieteellinen	Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

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Työn nimi — Arbetets titel — Title

On Hedging and Pricing of Options

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Lokakuu 2019	73 s.

Tiivistelmä — Referat — Abstract

In this thesis we cover some fundamental topics in mathematical finance and construct market models for the option pricing. An option on an asset is a contract giving the owner the right, but not the obligation, to trade the underlying asset for a fixed price at a future date. Our main goal is to find a price for an option that will not allow the existence of an arbitrage, that is, a way to make a riskless profit. We will see that the hedging has an essential role in this pricing. Both the hedging and the pricing are very import tasks for an investor trading at constantly growing derivative markets.

We begin our mission by assuming that the time parameter is a discrete variable. The advantage of this approach is that we are able to jump into financial concepts with just a small quantity of prerequisites. The proper understanding of these concepts in discrete time is crucial before moving to continuous-time market models, that is, models in which the time parameter is a continuous variable. This may seem like a minor transition, but it has a significant impact on the complexity of the mathematical theory.

In discrete time, we review how the existence of an equivalent martingale measure characterizes market models. If such measure exists, then market model does not contain arbitrages and the price of an option is determined by this measure via the conditional expectation. Furthermore, if the measure also unique, then all the European options (ones that can be exercised only at a predetermined time) are hedgeable in the model, that is, we can replicate the payoffs of those options with strategies constructed from other assets without adding or withdrawing capital after initial investments. In this case the market model is called complete. We also study how the hedging can be done in incomplete market models, particularly how to build risk-minimizing strategies. After that, we derive some useful tools to the problems of finding optimal exercise and hedging strategies for American options (ones that can be exercised at any moment before a fixed time) and introduce the Cox-Ross-Rubinstein binomial model to use it as a testbed for the methods we have developed so far.

In continuous time, we begin by constructing stochastic integrals respect to the Brownian motion, which is a stochastic component in our models. We then study important properties of stochastic integrals extensively. These help us comprehend dynamics of asset prices and portfolio values. In the end, we apply the tools we have developed to deal with the Black-Scholes model. Particularly, we use the Itô's lemma and the Girsanov's theorem to derive the Black-Scholes partial differential equation and further we exploit the Feynman-Kac formula to get the celebrated Black-Scholes formula.

Avainsanat — Nyckelord — Keywords

Hedging, Pricing, Option, Mathematical finance, Arbitrage, Martingale, Financial market model

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

University of Helsinki

Department of Mathematics and Statistics

Master's thesis

On Hedging and Pricing of Options

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October 16, 2019

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Chapter 1

Introduction

Consider the stocks of Finnish retailing conglomerate Kesko Corporation which are traded in the Helsinki Stock Exchange for the price of 58 euros per share at the time of writing. You are offered a contract (or an *option*) in which you have an opportunity to buy one stock of Kesko for the price of 60 euros at a future date. How much would you pay for this contract? Such an innocent question takes us to a long journey to the magnificent theory of the mathematical finance.

Let us try to unravel the problem. Our profit from the contract depends on the stock price so it would be beneficial to model the movements of the stock price somehow. As rational traders we do not want to pay too high a price for the contract and hence allowing seller to make a riskless profit at our expense. On the other hand, no-one is willing to sell us the contract with too low price. But what determines aforementioned "too high" and "too low" prices? The answer is often given by hedging.

Assume then, that we can replicate (or *hedge*) our profit from the contract with a portfolio constructed solely from other assets without adding or withdrawing capital after an initial investment. Let us assume further that we know the prices of those other assets. In this case the initial value of such replicating portfolio must coincide with the price of the contract. Otherwise there would be a possibility to make a riskless profit. A strategy that exploits this possibility is called an *arbitrage*.

Let us justify the claim we made. If the initial value of the replicating portfolio is higher, then we can short sell the portfolio and use the received money to buy the contract. If the initial value is lower, then we short sell the contract and buy the replicating portfolio. In both cases, we can make a positive profit without initial capital or the risk of losing money. This simplified principle describes the relationship between hedging and pricing and the idea will be carried throughout the thesis.

In this thesis we want to give pricing and hedging problems a rigorous (yet clear) mathematical treatment. We begin by restricting ourselves to financial market models with a discrete time variable. This allows us to dive into financial topics quickly without going into the rather demanding theory of stochastic integrals as in continuous-time models. It is important to understand financial notions in discrete time so that one's "forest's scenery is not obscured by the sight of the trees" when one moves on to mathematically more challenging continuous-time models.

1.1 Discrete time

We begin the chapter concerning discrete time by recalling the concepts of L^p-spaces and the conditional expectation. Then we directly move on to financial topics by mathematizing discrete-time financial market models. These models include assumptions of the underlying probability space, stochastic processes which represent asset prices, and the information structure. In the following sections, we study the properties that financial models may have, particularly arbitrage-freeness and completeness.

An important notion, concerning aforementioned properties, is self-financing strategy, that is, strategy which is build without adding or withdrawing capital after an initial investment. A model is called arbitrage-free if there does not exist self-financing strategies that can yield positive profit without initial capital or risk of loosing money. Further the model is called complete if every (European) claim can be replicated with a self-financing strategy.

Both arbitrage-freeness and completeness are characterized by the existence of an equivalent martingale measure, that is, a probability measure which has property that discounted asset price processes are martingales under this measure. This means that price processes, which are scaled by another asset price (called *numeraire*), are constant on average. We call the equivalent martingale measure *risk-neutral*, since it does not reward for risk-taking, that is, every asset price have the same expected value (equalling numeraire's expected value) under this measure despite the riskiness of the asset.

We will see that the existence of an equivalent martingale measure precludes the existence of an arbitrage. Furthermore, the existence of such measure is entirely determined by the arbitrage-freeness of a market model. This is called *the first fundamental theorem of asset pricing*. If the existing measure is also unique, then the market model is complete and vice versa. This claim is called *the second fundamental theorem of asset pricing*. These elegant theorems connect a rather abstract mathematical notion to practical financial concepts.

In the later section, we will introduce different types of options, particularly *European call* and *put options*. Aforementioned options give the buyer a right, but not an obligation, to trade the underlying asset for a fixed price at predetermined time. Prices of these mentioned options link to each other via so-called *put-call parity*, hence it suffices to find either one of the prices in order to know them both. In the one-period market model, the price of an option can be solved conveniently with a geometric analysis.

The equivalent martingale measure proves its usefulness once again when it comes to the pricing of options. We prove that the conditional expectation of the discounted payoff of an option, with respect to mentioned measure, yields such a price that arbitrage-freeness of a market model is preserved. If the market model is complete or the option is hedgeable, then the price is unique. If the option can not be replicated, then the prices belong to an open interval. Notice that there is always numeraire attached to an equivalent martingale measure. But we will see that it can be changed, by defining the *Radon-Nikodym derivative* a right way, so that the returned measure is another equivalent martingale measure.

Often market models are not complete, therefore we build ways hedge options, when the full replication is not possible. These techniques, particularly *risk-minimizing* methods, depend highly on the features of a underlying L^2-space. We will also familiarize ourselves with the *super-hedging*, that is, a way to surpass future claims with a self-financing strategy.

The American options generalize the European ones in a such way that they can be exercised at any moment before (and including) the fixed expiration time. This feature excites the problem of finding an optimal exercise strategy along with more complicated pricing and hedging problems. To tackle these problems, we introduce tools called the *Snell envelope*, that is, a supermartingale which dominates the payoff of an American option and the *Doob's decomposition*, that is, a way to decompose a stochastic process in to a martingale and predictable part.

The last topic we will cover in discrete time is called the *Cox-Ross-Rubinstein binomial market model*. In this simple model, we have two asset, one risky asset (say a stock) and one riskless asset (say a bank account). The bank account yields constant return, while the stock price has exactly two possible states in the next point of time: either it has gone up or down. Assuming reasonable conditions, we can find a unique equivalent martingale measure in this model, hence it is both arbitrage-free and complete. We will use this knowledge to derive the Cox-Ross-Rubinstein pricing formula for a European call option, which is the discrete-time analogue of the well-known *Black-Scholes formula*.

1.2 Continuous time

In the continuous time, we begin by introducing the *Brownian motion* and some of its well-known properties, for example, its *quadratic variation* up to time t is simply t. The Brownian motion will be a stochastic component in our continuous-time stock price processes. In the same section, we generalize concept of martingale to *local martingale* and even further to *semimartingale*. These notions appear, when we define stochastic integrals, that is, integrals with stochastic process as integrator.

Stochastic integrals and their properties are important, since they help us comprehend dynamics of asset prices and portfolio values. Particularly advantageous is to know, when a process defined by a stochastic integral is a local martingale or a (true) martingale and how to calculate the quadratic variation of such process. We will define and study stochastic integrals extensively. After that, we are ready to define *Itô processes* and express essential results from the Itô's calculus called the *Itô's formula* and the *stochastic integration by parts*.

An Itô process consists of *drift* (or the direction of movement) and *diffusion* (or the stochastic fluctuation of movement) terms, where the latter is defined by a stochastic integral. The Itô process is a (local) martingale if and only if it has a null drift. Itô's formula presents dynamics for a certain kind of function, that is given Itô processes as arguments. It can be seen as the stochastic calculus version of the chain rule, where the extra term comes from the non-zero quadratic variation.

In the following sections, we introduce two major theorems. The first one offers a way to change a probability measure so that we can eliminate the drift from the Itô process without affection for the diffusion term by changing the Brownian motion. This is called the *Girsanov's theorem* and it can help us find a risk-neutral measure that turns stock processes into (local) martingales and give the explicit stock price dynamics under this new measure.

The other theorem that we will motivate is called the *Feynman-Kac formula*. It constructs solution to deterministic partial differential equation with certain boundary condition by mixing stochastics into it. The link between deterministic problem and a stochastic process, in this theorem, results fundamentally from the Itô's formula. These type of boundary value problems are common in the mathematical finance, hence the Feynman-Kac formula, that originates from a physics problem, is useful to us.

In the final section, we introduce the *Black-Scholes model*. In this model we have two asset, one risky asset (say a stock) and one riskless asset (say a bank account). The bank account is assumed to compound constant interest continuously, while the stock price follows the *geometric Brownian motion*. We start by using Girsanov's theorem to get rid of the drift from the discounted stock price process. This stands for risk-neutral approach.

Then we use these risk-neutral stock price dynamics along with Itô's formula to derive the *Black-Scholes partial differential equation* for the value of the self-financing *Markovian strategy*, that is, a strategy which depends solely on the current value of the stock price process and the time parameter. We did this because, then we know that if a European option is replicable with a self-financing Markovian strategy, then the value of the replicating portfolio must satisfy the Black-Scholes partial differential equation and the solution to this yields the price for the option.

Fortunately, the Black-Scholes model is both arbitrage-free and complete (assumimg some extra restrictions to these definitions). This means that we can find a unique arbitrage-free price for a European option by solving the Black-Scholes partial differential equation with a boundary conditions given by the replication condition, that is, the terminal value of the replication portfolio must coincide with the payoff of the option. The solution to this boundary value problem is given by the Feynman-Kac formula. Using this knowledge, we derive the celebrated *Black-Scholes formula*, that is, formula for the arbitrage-free price of the European call option. In the end, we discuss some of the shortages of the Black-Scholes model and suggest further topics.

1.3 On bibliography

The most important sources of this thesis, in discrete time, are "Stochastic finance: An Introduction in Discrete Time" (2016) by Föllmer and Schied [10], "Mathematics of Financial Markets" (1999) by Elliot and Kopp [9], and "Financial Mathematics: Theory and Problems for Multi-period Models" (2012) by Pascucci and Runggaldier [16]. The corresponding sources, in continuous time, are "Arbitrage Theory in Continuous Time" (2004) by Björk [2], "Stochastic Processes" (2011) by Bass [1], and "PDE and Martingale Methods in Option Pricing" (2011) by Pascucci [15].

The following reviews are personal opinions of the writer of this thesis. It was surprising to discover that there exists so many books on the same topics but completely different level of rigour and generality. The best first book in (multi-period) discrete-time mathematical finance is Pascucci and Runggaldier [16] followed by Elliot and Kopp [9]. The first-mentioned is low on details and leaves gaps that have to be filled up by other sources afterwards. The latter does not have this problem but it is a bit harder to grasp.

The book by Föllmer and Schied [10] contains almost 600 pages and it can be considered as the cornucopia of the discrete-time mathematical finance. It might not be the most accessible book at first reading, since it contains a rather precise treatment of mathematical finance. But after familiarising oneself, for example, with [16] or [9], Föllmer and Schied turns into the tremendous source of information.

"The Mathematics of Arbitrage" (2006) by Delbaen and Schachermayer [7] and "Fundamentals and Advanced Techniques in Derivatives Hedging" by Bouchard [3] are great for mathematicians but rather abstract by nature. I would highly recommend reading something more intuitive (to understand the larger picture of finance) before entering these detail-oriented books. The latter deals with the most general case of discrete-time financial markets.

Both the discrete-time and continuous-time mathematical finance demand a strong background on the probability theory. In this thesis, the most important probability prerequisites for mathematical finance are given but obviously we are not able to give the exhaustive knowledge of probability theory. Good sources to strengthen one's scholarship are "Probability: Theory and Examples" (2010) by Durrett [8] and "Probability with Martingales" (1991) by Williams [19].

In continuous time, it was harder to find a suitable source. Overall the books were much more disorganized than discrete-time books. The book by Björk [2] is great on intuition but does not quench the thirst for mathematical details. I found the book by Pascucci [15] the most enjoyable to read, therefore the continuous-time part leans heavily on this source. The continuous-time mathematical finance requires a sound background on the stochastic calculus. I would recommend "Introduction to Stochastic Calculus for Finance: A New Didactic Approach" (2007) by Dieter Sondermann for the first book in stochastic calculus, even though this is not listed in the bibliography at the end of the thesis.

Although the lectures and notes in the University of Helsinki are not mentioned as a direct reference in the bibliography, those have had a big influence on the thesis. Particularly influential were lectures notes on financial economics by Harri Nyrhinen, lectures on mathematical finance and discussions after and during the lectures by Dario Gasbarra, and lecture notes on mathematical finance by Tommi Sottinen.

This thesis contains constant balancing between rigour and clarity. Our main goal is to yield applicable results to the problems of pricing and hedging of options. Therefore some irrelevant details are omitted but references to these details are given. Nevertheless the self-contained theory is included to support the results.

Chapter 2

Discrete-time financial market

In this chapter, we assume that the time parameter is a discrete variable with distinct values $0, 1, 2, \ldots, T$, where $T \in \mathbb{N}$.

2.1 Some probability preliminaries

Before going to financial applications, we recall some notions from probability theory that we will need later. Other preliminaries will be introduced as we progress through thesis. We restrict ourselves to real-valued random variables and stochastic processes unless otherwise stated.

Definition 2.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then for every $p \in [1, \infty]$ we denote

$$L^{(p)}(\Omega, \mathcal{F}, \mathbb{P}) := \{X : \Omega \to \mathbb{R} \mid X \text{ is } \mathcal{F}\text{-measurable}, \|X\|_p < \infty\}, \tag{2.1.2}$$

where the *p-norm* $\|\cdot\|_p$ of random variable X is defined by

$$\|X\|_p := \mathbb{E}(|X|^p)^{\frac{1}{p}} = \left(\int_\Omega |X(\omega)|^p d\mathbb{P}(\omega) \right)^{\frac{1}{p}}, \tag{2.1.3}$$

when $p \in [1, \infty)$ and

$$\|X\|_p := \inf\{M \geq 0 \mid \mathbb{P}(|X| > M) = 0\} \tag{2.1.4}$$

when $p = \infty$. Further X is called *essentially bounded* if $\|X\|_\infty < \infty$. $\Diamond$

Technically $\|\cdot\|_p$ is only a *seminorm* in $L^{(p)}(\Omega, \mathcal{F}, \mathbb{P})$. If we want a (true) norm, we need to be little more explicit. Let us denote

$$\mathcal{X}_0 = \{X : \Omega \to \mathbb{R} \mid X \text{ is } \mathcal{F}\text{-measurable}, X = 0 \ \mathbb{P}\text{-almost surely}\} \tag{2.1.5}$$

and define equivalence class $[X] := X + \mathcal{X}_0$ for $X \in L^{(p)}(\Omega, \mathcal{F}, \mathbb{P})$. Thus for every $Y \in [X]$ it holds that $Y = X$ $\mathbb{P}$-almost surely. Now we are ready for conclusive definition.

Definition 2.1.6. For every $p \in [1, \infty]$ we denote

$$L^p(\Omega, \mathcal{F}, \mathbb{P}) := \{[X] := X + \mathcal{X}_0 \mid X \in L^{(p)}(\Omega, \mathcal{F}, \mathbb{P})\} \tag{2.1.7}$$

and define $\|[X]\|_p := \|X\|_p$ and $\mathbb{E}([X]) := \mathbb{E}(X)$ for $[X] \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. $\diamondsuit$

We will often use notation $L^p(\mathbb{P})$ or even L^p for $L^p(\Omega, \mathcal{F}, \mathbb{P})$ if some of $\Omega, \mathcal{F}$, and $\mathbb{P}$ are clear from the context. Recall that p-norm $\|\cdot\|_p$ is a true norm in L^p and that L^p has some useful features. For example, for every $p \in [1, \infty]$ we have that L^p equipped with p-norm is *complete* normed space (or *Banach space*), that is, every Cauchy-sequence converges. Particularly we will use L^2 space which is complete inner product space (or *Hilbert space*) when it is equipped with inner product

$$\langle [X], [Y] \rangle := \mathbb{E}(XY) = \int_\Omega X(\omega)Y(\omega)d\mathbb{P}(\omega) \text{ for } [X], [Y] \in L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.1.8}$$

We notice that inner product induces norm to L^2, since $\|[X]\|_2 = \sqrt{\langle [X], [X] \rangle}$. Additionally, we will denote the space (of equivalence classes) of random variables by

$$L^0(\Omega, \mathcal{F}, \mathbb{P}) := \{[X] := X + \mathcal{X}_0 \mid X : \Omega \to \mathbb{R},\ X \text{ is } \mathcal{F}\text{-measurable}\}. \tag{2.1.9}$$

From now on, we will leave out square brackets $[\cdot]$, when we refer to elements of L^p or L^0 spaces. So instead of talking about equivalence classes, we talk about underlying random variables for simplicity. Also for multidimensional random variable $X = (X^1, \dots, X^d)$ we use $X \in L^p$ to indicate that $X^i \in L^p$ for every $i = 1, \dots, d$. In the next definition we denote by $\mathbb{1}_A : \Omega \to \{0, 1\}$ the indicator random variable, such that $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{1}_A(\omega) = 0$ otherwise.

Definition 2.1.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{G} \subseteq \mathcal{F}$ a sub-sigma-algebra of $\mathcal{F}$. A *Conditional expectation* of random variable $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ given $\mathcal{G}$, denoted by $\mathbb{E}(X|\mathcal{G})$, is ($\mathcal{G}$-measurable) random variable in $L^1(\Omega, \mathcal{G}, \mathbb{P})$ which satisfies

$$\mathbb{E}(\mathbb{1}_A \mathbb{E}(X|\mathcal{G})) = \mathbb{E}(\mathbb{1}_A X) \quad \text{for each } A \in \mathcal{G} \tag{2.1.11}$$

$\diamondsuit$

The conditional expectation has lots of useful properties. Next lemma covers the ones we will need later. For the proof of existence of conditional expectation and its properties, see for example Chapter 9 of [19].

Lemma 2.1.12. *(Properties of conditional expectation) Let us assume that* $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and* $a, b \in \mathbb{R}$, *then (using the notation of Definition 2.1.10) we have (*$\mathbb{P}$*-almost surely)*

(i) linearity: $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$,

(ii) independency property: if X *is independent of* $\mathcal{G}$, *then* $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$,

(iii) "taking out what is known": if X *is* $\mathcal{G}$*-measurable, then* $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$, *particularly* $\mathbb{E}(X|\mathcal{G}) = X$,

(iv) tower property: if $\mathcal{H}$ *is a sub-sigma-algebra of* $\mathcal{G}$, *then* $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$, *particularly* $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

We will later introduce other probability measures. Notation $\mathbb{E}_{\widehat{\mathbb{P}}}$ indicates that the expectation is to be taken under the measure $\widehat{\mathbb{P}}$. A plain symbol $\mathbb{E}$ means that expectation is to be taken under the initial probability measure $\mathbb{P}$.

2.2 Market dynamics and information

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that sample space Ω has a finite number of elements and that $\mathcal{F}$ is the power set of Ω with $\mathbb{P}(\{\omega\}) > 0$ for any $\omega \in \Omega$. The prices of assets (for example stock prices) change in time and can be considered as stochastic processes. Elements $\omega \in \Omega$ are called *scenarios* since they correspond to different scenarios of the possible changes of the asset prices. The initial probability measure $\mathbb{P}$ is called *objective* or *physical* measure. Let us assume that $t_0 < t_1 < \cdots < t_N$ are real-valued and denote by

$$\mathbb{T} = \{t_0, t_1, \dots, t_N\},$$

the set of the *trading times*, where $N \in \mathbb{N}$ determines the number of trading periods. Without loss of generality, we assume that $\mathbb{T} = \{0, 1, \dots, T\}$, where $t_0 = 0$ can be interpreted as today's date and $t_N = T \in \mathbb{N}$ as the expiration date of a *derivative* (more on this later).

We suppose that the market consists of a non-risky asset (bank account) S^0 and $d \in \mathbb{N}$ number of risky assets (stocks) $S^1, \dots, S^d$. Each $S^i = \{S^i_t \mid t \in \mathbb{T}\}$ is a positively real-valued stochastic process, where random variable $S^i_t > 0$ denotes the price of an asset at time $t \in \mathbb{T}$ for every $i = 0, 1, \dots, d$. We assume that the non-risky asset has the following deterministic dynamics

$$S^0_0 = 1 \quad \text{and} \quad \Delta S^0_t = r_t S^0_{t-1} \text{ for } t \in \mathbb{T} \setminus \{0\}, \tag{2.2.1}$$

where $\Delta X_t = X_t - X_{t-1}$ and $r_t > -1$ denotes risk-free interest rate in the period $(t-1, t]$. In a similar fashion, the risky assets have the following stochastic dynamics

$$S^i_0 > 0 \quad \text{and} \quad \Delta S^i_t = R^i_t S^i_{t-1} \text{ for } t \in \mathbb{T} \setminus \{0\}, \tag{2.2.2}$$

where R^i_t is a random variable that represents the rate of return of the asset i in the period $(t-1, t]$. The dynamics of the risky asset can be written in a demonstrative form $S^i_t = (1 + R^i_t)S^i_{t-1}$. We could also calculate the rate of return $R^i_t = (S^i_t - S^i_{t-1})/S^i_{t-1}$, provided that we knew the prices of the asset i at times $t-1$ and t.

We make a natural assumption that the investor's decisions at moment $t \in \mathbb{T}$ can be based on information available up to the moment t and not the future information. Furthermore, we assume that the information is non-degreasing in time so that investors learn without forgetting. The information structure available to the investors is given by an increasing and finite sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_T = \mathcal{F}$ of sub-sigma-algebras of $\mathcal{F}$. In the information structure we assume $\mathcal{F}_0$ to be trivial, that is, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We call an increasing family of sigma-algebras a *filtration* $\mathbb{F} = \{\mathcal{F}_t \mid t \in \mathbb{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and a probability space equipped with the filtration $\mathbb{F}$ a *stochastic basis* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

A stochastic process $X = \{X_t \,|\, t \in \mathbb{T}\}$ is called *adapted* (to the filtration $\mathbb{F}$), if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{T}$. This means that, if X is adapted, then the value $X_t(\omega)$ is known to us at time t. We assume every price process S^i is adapted and thus every the rate of return process R^i is adapted. Let us denote $R_t = (R^1_t, \dots, R^d_t)$ and $S_t = (S^0_t, S^1_t, \dots, S^d_t)$. Since in the market, based on (2.2.2), the sequence R_t will be the only source of randomness, we assume that $\mathcal{F}_t$ is generated by R_t, so that $\mathcal{F}_t = \mathcal{F}^R_t = \sigma\{R_k \mid k \le t\}$ for $t \in \mathbb{T} \setminus \{0\}$. By the bijective correspondence between the processes R^i and S^i, we have $\mathcal{F}_t = \mathcal{F}^R_t = \mathcal{F}^S_t$. Finally we have defined a *market model* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, \mathbb{F}, S)$.

Remark 2.2.3. At the beginning of the section, we assumed that we are operating in the finite probability space, that is, sample space Ω has finitely many elements. Consequently, every $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, \infty]$ contains the same random variables. Furthermore, in the market model defined above, the price of an asset i at time t, explicitly $S_t^i : \Omega \to \mathbb{R}$, belongs to $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ for every $p \in [1, \infty]$. Although this discrete time market model is not the most general setting, the results in the following sections usually hold in the more general discrete time models. We will use the notion of L^p spaces occasionally anyway to remind us of the generality of results. The advantage of our market model is that, it provides intuition and clarity for the mathematics of financial concepts. $\triangle$

Example 2.2.4. Let us define two-period market model with two assets, stock S and bank account B. Now $\mathbb{T} = \{0, 1, 2\}$ and we define

$$\Omega = \{(\omega_1, \omega_2) \in \mathbb{R}^2 \mid \omega_1, \omega_2 \in \{0, 1\}\} = \{(0,0), (1,0), (0,1), (1,1)\},$$

where we assume that $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. Let us suppose that the bank account yields with constant interest rate $r = 1/10$, so that B has dynamics

$$B_t = (1+r)^t, \quad t = 0, 1, 2$$

and assume that the stock price has the following dynamics

$$S_0 = 150, \quad S_1(\omega_1) = [1 + R(\omega_1)]S_0 \quad \text{and} \quad S_2(\omega_1, \omega_2) = [1 + R(\omega_2)]S_1(\omega_1),$$

with the rate of return given by

$$R(\omega_t) = \begin{cases} \dfrac{2}{5}, & \omega_t = 1 \\ -\dfrac{1}{5}, & \omega_t = 0 \end{cases}$$

for $t = 1, 2$. Figure 2.1 visualizes the evolution of the stock price process. Each node in the tree corresponds to a stock price at given time and "the state of the world". Filtration $\mathbb{F} = \{\mathcal{F}_t \mid t = 0, 1, 2\}$ is given by

$$\begin{cases} \mathcal{F}_0 & = \{\emptyset, \Omega\} \\ \mathcal{F}_1 & = \big\{ \{\omega \in \Omega \mid S_1(\omega) = 120\}, \{\omega \in \Omega \mid S_1(\omega) = 210\} \big\} \cup \mathcal{F}_0 \\ & = \big\{ \{(0,0), (0,1)\}, \{(1,0), (1,1)\} \big\} \cup \mathcal{F}_0 \\ \mathcal{F}_2 & = 2^\Omega, \end{cases}$$

where $\mathcal{F}_2$ is the power set of Ω, that is, it contains all the subsets of Ω so in this case 16 different subsets. Intuitively $\mathcal{F}_0$ does not contain any information about the market, $\mathcal{F}_1$ contains information whether the stock price has gone up or down in the first time period and, $\mathcal{F}_2$ contains all the information, that can be received from the market. $\triangle$

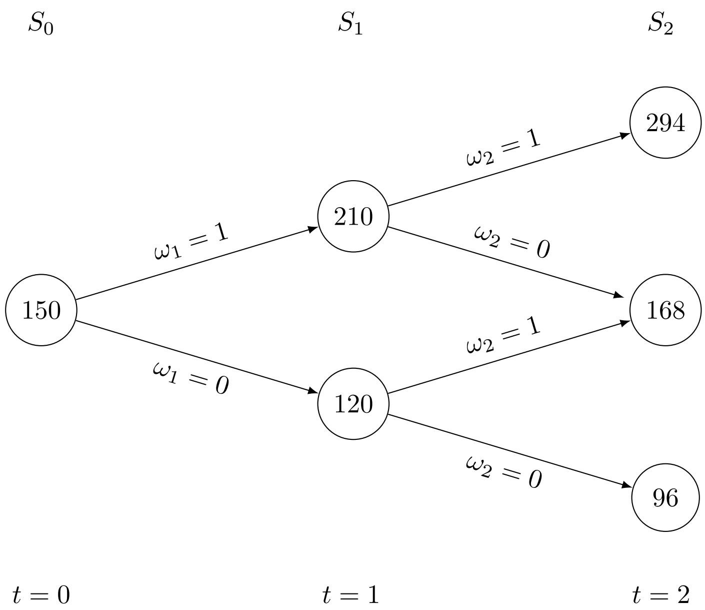

Figure 2.1: The possible paths of the stock price process form a tree-shaped structure.

2.3 Self-financing strategies and discounting

Consider $\mathbb{R}^{d+1}$-valued stochastic process $\theta = \{(\theta_t^0, \theta_t^1, \ldots, \theta_t^d) \mid t \in \mathbb{T}\}$, where θ_t^i represents the amount of asset S^i hold during the period $(t-1, t]$. Aforementioned vector-valued process θ is called a *strategy* (or a *portfolio*) and the *value of the portfolio* θ at time t (precisely just after the asset values have changed to S_t) is given by

$$V_t^\theta = \theta_t \cdot S_t := \sum_{i=0}^{d} \theta_t^i S_t^i \quad \text{for } t \in \mathbb{T}, \tag{2.3.1}$$

where we used notation $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$ for price vector at time t and notation $\theta_t = (\theta_t^0, \theta_t^1, \ldots, \theta_t^d)$ for strategy in time interval $(t-1, t]$. The value V_0^θ is the investor's initial investment. The investor chooses his/hers time t portfolio θ_t as soon as the stock prices at time $t-1$ are known. Then he/she holds this portfolio the time interval $(t-1, t]$. Therefore we require that strategy is a *predictable* stochastic process, that is, $X = \{X_t \mid t \in \mathbb{T}\}$ is predictable, if X_{t+1} is $\mathcal{F}_t$-measurable for every $t \in \mathbb{T} \setminus \{T\}$.

Remark 2.3.2. Note that we allow negative values for θ_t^i. We interpret negative amount of asset as short-selling or borrowing of assets. △

Definition 2.3.3. A strategy θ is *self-financing* if it satisfies

$$\theta_{t+1} \cdot S_t = \theta_t \cdot S_t, \tag{2.3.4}$$

for every $t \in \mathbb{T} \setminus \{T\}$. ◊

From (2.3.1) we know that $\theta_t \cdot S_t$ is the value of the portfolio at time t. So in the self-financing strategy, the new portfolio θ_{t+1} (for time interval $(t, t+1]$) is created with investor's capital from the portfolio at time t without adding new capital outside of the portfolio and without withdrawing capital from the portfolio.

Remark 2.3.5. Since there is no information available before time $t = 0$, we set $\theta_0 = \theta_1$. △

Let us assume that θ is a self-financing strategy. Hence

$$\begin{aligned} \Delta V_t^\theta &= \theta_t \cdot S_t - \theta_{t-1} \cdot S_{t-1} \\ &= \theta_t \cdot S_t - \theta_t \cdot S_{t-1} \\ &= \theta_t \cdot \Delta S_t. \end{aligned} \tag{2.3.6}$$

We define the *gain process* associated with θ by setting

$$G_0^\theta = 0 \quad \text{and} \quad G_t^\theta = \sum_{u=1}^{t} \theta_u \cdot \Delta S_u \text{ for } t \in \mathbb{T} \setminus \{0\}. \tag{2.3.7}$$

Now we see that θ is self-financing if and only if

$$V_t^\theta = V_0^\theta + G_t^\theta \tag{2.3.8}$$

for all $t \in \mathbb{T}$.

For a rational investor money available today is worth more than the identical sum in the future because money has potential to grow in value over a given period of time. If we want to compare two sums from the different moments of time, we have to take into account this time value of money. Let us introduce *discounted price* of the asset i and *discounted value* of the portfolio θ which are defined by

$$\widetilde{S}_t^i := \frac{S_t^i}{S_t^0} \quad \text{and} \quad \widetilde{V}_t^\theta := \theta_t \cdot \widetilde{S}_t \tag{2.3.9}$$

for $t \in \mathbb{T}$, respectively. Furthermore, if θ is self-financing, then (2.3.6) and (2.3.8) yield

$$\widetilde{V}_t^\theta = V_0^\theta + \sum_{u=1}^{t} \theta_u \cdot \Delta \widetilde{S}_u = V_0^\theta + \widetilde{G}_t^\theta \tag{2.3.10}$$

for all $t \in \mathbb{T}$. In equations (2.3.9) the asset S^0 is called a *numeraire*. Numeraire is an asset in terms of whose price the relative prices of all other asset are expressed. In other words, other prices are normalized by the price of the numeraire asset. We could have taken any of the assets as a numeraire as long as the price of chosen asset is strictly positive. Choosing the right numeraire can make computations much easier.

2.4 Arbitrage and martingale measures

An arbitrage is a strategy that can yield positive profit without initial capital or risk of losing money. Loosely speaking the arbitrage creates money out of thin air. With rational investors in the market, there should not exist arbitrages. One can argue that if arbitrage existed, then many of investors would exploit it and thus the prices of assets would correct themselves, by the law of supply and demand, in a such way that the arbitrage would vanish. This reasoning restricts our attention to markets with no arbitrages.

Definition 2.4.1. Self-financing and predictable strategy θ is an *arbitrage* if the following conditions hold

$$V_0^\theta = 0, \quad \mathbb{P}(V_T^\theta \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(V_T^\theta > 0) > 0. \tag{2.4.2}$$

We say that a market model is *arbitrage-free* if there does not exist an arbitrage. ◊

Next we will start to lay the foundation of an important concept called equivalent martingale measure. We will see that the existence of such measure is closely connected with the existence of an arbitrage.

Definition 2.4.3. A stochastic process $M = \{M_t \mid t \in \mathbb{T}\}$ is called a *martingale* (with respect to $(\mathbb{F}, \mathbb{P})$) if $M_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in \mathbb{T}$ and

$$\mathbb{E}(M_t|\mathcal{F}_{t-1}) = M_{t-1} \quad \text{for every } t \in \mathbb{T} \setminus \{0\}. \tag{2.4.4}$$

A *supermartingale* is defined similarly, except that the last condition is replaced by $\mathbb{E}(M_t|\mathcal{F}_{t-1}) \leq M_{t-1}$, and a *submartingale* is defined by replacing it by $\mathbb{E}(M_t|\mathcal{F}_{t-1}) \geq M_{t-1}$. ◊

Note that for a martingale M it holds that $\mathbb{E}(\Delta M_t|\mathcal{F}_{t-1}) = 0$ for all $t \in \mathbb{T} \setminus \{0\}$. Thus, by the law of total expectation, $\mathbb{E}(\Delta M_t) = 0$ and hence $\mathbb{E}(M_t) = \mathbb{E}(M_{t-1})$ for all $t \in \mathbb{T} \setminus \{0\}$, so that a martingale is "constant on average". Similarly, a supermatingale decreases, and a submartingale increases on average. If we assume that gambler's wealth process is a martingale, then we can interpret that gambler is playing a "fair" game. A supermartingale would model "unfavourable" and a submartingale "favourable" game.

Furthermore, for every $t \in \mathbb{T} \setminus \{T\}$, we have

$$M_t = \mathbb{E}(M_{t+1}|\mathcal{F}_t) = \mathbb{E}(\mathbb{E}(M_{t+2}|\mathcal{F}_{t+1})|\mathcal{F}_t) = \mathbb{E}(M_{t+2}|\mathcal{F}_t),$$

by the tower property of conditional expectation. We could continue the procedure above (until we reach time T in the expectation) and hence we have the following useful result: $M_u = \mathbb{E}(M_t|\mathcal{F}_u)$ for every $u \leq t$, when $u, t \in \mathbb{T}$. We also need the following lemma of martingales later:

Lemma 2.4.5. *Let $c \in \mathbb{R}$ be a constant and let also M^1 and M^2 be martingales (with respect to $(\mathbb{F}, \mathbb{P})$), then $c + M^1 := \{c + M_t^1 \mid t \in \mathbb{T}\}$ and $M^1 + M^2 := \{M_t^1 + M_t^2 \mid t \in \mathbb{T}\}$ are martingales (with respect to $(\mathbb{F}, \mathbb{P})$).*

Proof. The claim follows immediately from the linearity of conditional expectation. □

Definition 2.4.6. Let $M = \{M_t \mid t \in \mathbb{T}\}$ be a (super)martingale and $\phi = \{\phi_t \mid t \in \mathbb{T}\}$ be a predictable stochastic process. Process $X = \phi \bullet M$ given by

$$X_0 = 0 \quad \text{and} \quad X_t = \sum_{u=1}^{t} \phi_u \Delta M_u \quad \text{for } t \in \mathbb{T} \setminus \{0\} \tag{2.4.7}$$

is the *(super)martingale transformation* of M by ϕ. ◊

The martingale transform $\phi \bullet M$ is the discrete analogue of the stochastic integral $\int \phi \, dM$. We can think ϕ_u as stake on game at time u. Since ϕ is predictable, the value of ϕ_u can be decided based on the information up to (and including) time $u-1$. The return of the game in time interval $(u-1, u]$ is $\phi_u(M_u - M_{u-1})$ $(= \phi_u \Delta M_u)$ and the total return up to time t is

$$(\phi \bullet M)_t = \phi_1 \Delta M_1 + \phi_2 \Delta M_2 + \ldots + \phi_t \Delta M_t = \sum_{u=1}^{t} \phi_u \Delta M_u.$$

Proposition 2.4.8. *("We can't beat the system with a finite capital") Let process ϕ be predictable and bounded, so that for some $K \in [0, \infty)$, $|\phi_u(\omega)| \leq K$ for every $u \in \mathbb{T}$ and $\omega \in \Omega$.*

(i) If M is a martingale, then $\phi \bullet M$ is a martingale.

(ii) If M is a supermartingale and ϕ is non-negative, then $\phi \bullet M$ is a supermartingale.

Proof. Write $X = \phi \bullet M$. Since ϕ is bounded and $\mathcal{F}_{t-1}$-measurable,

$$\mathbb{E}(X_t - X_{t-1} | \mathcal{F}_{t-1}) = \phi_t \mathbb{E}(M_t - M_{t-1} | \mathcal{F}_{t-1}) = 0 \text{ (and } \leq 0 \text{ for a supermartingale).}$$

□

Consider discounted gain process $\widetilde{G}_t^\theta$ of self-financing strategy θ. We can write the process as

$$\widetilde{G}_t^\theta = \sum_{u=1}^{t} \theta_u \cdot \Delta \widetilde{S}_u = \sum_{i=0}^{d} \sum_{u=1}^{t} \theta_u^i \Delta \widetilde{S}_u^i = \sum_{i=0}^{d} (\theta^i \bullet \widetilde{S}^i)_t. \tag{2.4.9}$$

Thus if $\widetilde{S}$ is martingale under some probability measure $\mathbb{Q}$, then the discounted gain process of a self-financing strategy is finite sum of martingales transforms and hence a martingale itself. Recall that $\widetilde{G}_0^\theta = 0$, therefore $\mathbb{E}_\mathbb{Q}(\widetilde{G}_t^\theta) = \mathbb{E}_\mathbb{Q}(\widetilde{G}_0^\theta) = 0$, since $\widetilde{G}^\theta$ is a martingale. Recall also the equation (2.3.10) that is $\widetilde{V}_t^\theta = V_0^\theta + \widetilde{G}_t^\theta$, thus $\mathbb{E}_\mathbb{Q}(\widetilde{V}_T^\theta) = \mathbb{E}_\mathbb{Q}(V_0^\theta)$.

This **precludes the existence of an arbitrage** with respect to probability measure $\mathbb{Q}$: if $V_0^\theta = 0$ and $\mathbb{Q}(V_T^\theta \geq 0) = 1$, but $\mathbb{E}_\mathbb{Q}(\widetilde{V}_T^\theta) = 0$, it follows that $\mathbb{Q}(V_T^\theta = 0) = 1$. This remains true for $\mathbb{P}$, if these two probability measures ($\mathbb{P}$ and $\mathbb{Q}$) agree on which events have probability zero. If such measure $\mathbb{Q}$ can be found, then no self-financing and predictable strategy θ can be an arbitrage, thus the market model is arbitrage-free.

Definition 2.4.10. Probability measures $\widehat{\mathbb{P}}$ and $\mathbb{P}$, on the same measurable space $(\Omega, \mathcal{F})$, are called *equivalent* if and only if

$$\widehat{\mathbb{P}}(A) = 0 \iff \mathbb{P}(A) = 0 \quad \text{for every } A \in \mathcal{F}.$$

We use notation $\widehat{\mathbb{P}} \sim \mathbb{P}$ for equivalent measures. ◊

Remark 2.4.11. If sample space Ω has finite (or countable) amount of elements then $\widehat{\mathbb{P}} \sim \mathbb{P}$ if and only if $\widehat{\mathbb{P}}(\{\omega\}) = 0 \iff \mathbb{P}(\{\omega\}) = 0$ for every $\omega \in \Omega$. Recall that we exclude from the consideration ω such that $\mathbb{P}(\{\omega\}) = 0$. So $\widehat{\mathbb{P}} \sim \mathbb{P}$ if and only if $\widehat{\mathbb{P}}(\{\omega\}) > 0$ for every $\omega \in \Omega$. △

Definition 2.4.12. An *equivalent martingale measure* with numeraire S^0 is a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\widetilde{S}_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q})$ for all $t \in \mathbb{T}$,

$$\mathbb{Q} \sim \mathbb{P} \quad \text{and} \quad \widetilde{S}_{t-1} = \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_t | \mathcal{F}_{t-1}) \quad \text{for all } t \in \mathbb{T} \setminus \{0\}, \tag{2.4.13}$$

that is, $\mathbb{Q}$ is such that discounted price process $\widetilde{S}$ is $(\mathbb{F}, \mathbb{Q})$-martingale.

We denote the set of all equivalent martingale measures (with essentially bounded density) by $\mathcal{M}^b$. We will deal with the notion of density later and often do not mention about the density, when referring to these measures. $\Diamond$

An equivalent martingale measure is often called *risk neutral measure*. The latter name comes from the fact that such measure does not reward for risk-taking. The expected return is same (the return of the non-risky asset) for every asset under this measure despite the riskiness of an asset. This does **not** mean that we believe that we live in a risk neutral world but we will still carry out the computations as if we lived. The advantage of such an approach comes when we valuate options since every investor agrees on the valuation regardless of their attitude towards risk and the price does not allow arbitrages.

Example 2.4.14. Let us find out if market model defined in Example 2.2.4 is arbitrage-free by searching the equivalent martingale measure $\mathbb{Q}$. Recall that

$$\Omega = \{(0,0), (1,0), (0,1), (1,1)\}$$

and $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$ so we denote conveniently

$$q_{ij} = \mathbb{Q}(\{(i,j)\}) > 0 \quad \text{for } i, j = 0, 1.$$

Since $\mathbb{Q}$ must be probability measure, we have $\mathbb{Q}(\Omega) = 1$, that is

$$q_{00} + q_{10} + q_{01} + q_{11} = 1.$$

Recall that $\mathcal{F}_1 = \{\emptyset, A_0, A_1, \Omega\}$, where

$$\begin{cases} A_0 = \{S_1 = 120\} = \{\omega \in \Omega \,|\, S_1(\omega) = 120\} = \{(0,0), (0,1)\} \\ A_1 = \{S_1 = 210\} = \{(1,0), (1,1)\}. \end{cases}$$

Hence $\mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2 | \mathcal{F}_1)$ is a random variable such that

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2 | \mathcal{F}_1)(\omega) = \begin{cases} \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2 | A_0), & \omega \in A_0 \\ \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2 | A_1), & \omega \in A_1, \end{cases}$$

where on the right hand side the expectations are calculated with respect to measure $\mathbb{Q}(\,\cdot\,|A_i)$, defined by

$$\mathbb{Q}(\{\omega\}|A_i) = \frac{\mathbb{Q}(\{\omega\} \cap A_i)}{\mathbb{Q}(A_i)}, \ \forall \omega \in \Omega$$

for $i = 0, 1$. Now if there exists equivalent martingale measure (with numeraire B), then

$$\begin{cases} \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_1) = \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_1|\mathcal{F}_0) = \widetilde{S}_0 \\ \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2) = \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2|\mathcal{F}_0) = \widetilde{S}_0 \\ \mathbb{E}_{\mathbb{Q}}(\widetilde{S}_2|\mathcal{F}_1)(\omega) = \widetilde{S}_1(\omega), \text{ for } \omega \in \Omega. \end{cases}$$

We can take out the deterministic discounting factor and use the stock prices shown in the tree in Example 2.2.4 to calculate expectations so the equations above yield four equations depending on q_{ij}:

$$\begin{cases} \frac{10}{11} \cdot [210(q_{10} + q_{11}) + 120(q_{01} + q_{00})] = 150 \\ \frac{100}{121} \cdot [294 q_{11} + 168(q_{01} + q_{10}) + 96 q_{00}] = 150 \\ \frac{100}{121} \cdot [168 \frac{q_{01}}{q_{01}+q_{00}} + 96 \frac{q_{00}}{q_{01}+q_{00}}] = \frac{10}{11} \cdot 120 \\ \frac{100}{121} \cdot [294 \frac{q_{11}}{q_{10}+q_{11}} + 168 \frac{q_{10}}{q_{10}+q_{11}}] = \frac{11}{10} \cdot 210. \end{cases}$$

We can choose any 3 out of these 4 equations along with aforementioned $\sum_{i,j=0}^{1} q_{ij} = 1$ and solve for q_{ij}. We find out that there exists unique symmetric equivalent martingale measure $\mathbb{Q}$ (with numeraire B) satisfying

$$q_{00} = q_{10} = q_{01} = q_{11} = \frac{1}{4}.$$

Thus the market model in Example 2.2.4 is arbitrage-free. △

It can be proven that the existence of an equivalent martingale measure does not only imply the absence of arbitrage but also the reverse implication is true. Next important theorem states this fact with technical details considering *Radon-Nikodymin derivative* (or *density*) which is introduced in Section 2.11.

Theorem 2.4.15. *(First fundamental theorem of asset pricing) A discrete-time market model is arbitrage-free if and only if there exists at least one equivalent martingale measure $\mathbb{Q}$ with essentially bounded density, that is, $d\mathbb{Q}/d\mathbb{P} \in L^\infty(\mathbb{P})$.*

Proof. "$\Leftarrow$": In the earlier discussion we already concluded that the existence of equivalent martingale measure ensures that our market model is arbitrage-free.
"$\Rightarrow$": In this direction the proof is more technical and we will skip it. Nevertheless, we will mention that in general probability space the proof of the theorem requires well-known result from functional analysis called the *Hahn-Banach separation theorem*. The general version of the proof can be found, for example, from the first chapter of [3]. In a finite probability space, one can use the special case of the Hahn-Banach separation theorem called the *separating hyperplane theorem*. See, for example, the proof of Theorem 3.2.2 in [9]. □

With respect to risk neutral measure, the processes of the discounted prices of assets are martingales. The next proposition shows that this holds also for any self-financing and predictable strategy.

Proposition 2.4.16. *Let $\mathbb{Q}$ be an equivalent martingale measure and θ a predictable and self-financing strategy with value V^θ. Then we have*

$$\widetilde{V}_{t-1}^\theta = \mathbb{E}_\mathbb{Q}(\widetilde{V}_t^\theta | \mathcal{F}_{t-1}) \quad \textit{and particularly} \quad V_0^\theta = \mathbb{E}_\mathbb{Q}(\widetilde{V}_t^\theta) \tag{2.4.17}$$

for every $t \in \mathbb{T} \setminus \{0\}$.

Proof. For a self-financing strategy θ the equation (2.3.6) states that $\widetilde{V}_t^\theta = \widetilde{V}_{t-1}^\theta + \theta_t \cdot \Delta \widetilde{S}_t$. Taking the conditional expectation, we get

$$\mathbb{E}_\mathbb{Q}(\widetilde{V}_t^\theta | \mathcal{F}_{t-1}) = \widetilde{V}_{t-1}^\theta + \theta_t \cdot \underbrace{\mathbb{E}_\mathbb{Q}(\Delta \widetilde{S}_t | \mathcal{F}_{t-1})}_{=0} = \widetilde{V}_{t-1}^\theta$$

since $\widetilde{S}$ is $\mathbb{Q}$-martingale and θ_t is $\mathcal{F}_{t-1}$-measurable. □

Corollary 2.4.18. *In an arbitrage-free market, if two predictable self-financing strategies θ and ϕ have the same final value $V_T^\theta = V_T^\phi$ almost surely, then they are also such that $V_t^\theta = V_t^\phi$ almost surely for every $t \in \mathbb{T}$.*

Proof. Since the market is arbitrage-free, Theorem 2.4.15 states that there exists an equivalent martingale measure $\mathbb{Q}$. Thus by Proposition 2.4.16

$$\widetilde{V}_t^\theta = \mathbb{E}_\mathbb{Q}(\widetilde{V}_T^\theta | \mathcal{F}_t) = \mathbb{E}_\mathbb{Q}(\widetilde{V}_T^\phi | \mathcal{F}_t) = \widetilde{V}_t^\phi$$

almost surely for every $t \in \mathbb{T}$. □

2.5 Local arbitrage

In earlier sections, we considered only times 0 and T as we dealt with the definition of arbitrage. In this section we will observe the intuitive matter that arbitrage-free markets are arbitrage-free for every time period $[t, t+1]$ before T. First we formulate arbitrage-free condition explicitly.

Definition 2.5.1. A *no arbitrage condition* denoted by (NA) holds if for every self-financing and predictable strategies θ for which $V_0^\theta = 0$ the following holds:

$$\widetilde{V}_T^\theta \geq 0 \quad \mathbb{P}\text{-a.s.} \quad \Rightarrow \quad \widetilde{V}_T^\theta = 0 \quad \mathbb{P}\text{-a.s.}$$

◊

In a similar fashion we define no arbitrage condition in one period.

Definition 2.5.2. A *local no arbitrage condition* denoted by $(\mathrm{NA})_t$ for $t \in \mathbb{T} \setminus \{T\}$ is defined by:

$$\xi \cdot \Delta \widetilde{S}_{t+1} \geq 0 \quad \mathbb{P}\text{-a.s.} \quad \Rightarrow \quad \xi \cdot \Delta \widetilde{S}_{t+1} = 0 \quad \mathbb{P}\text{-a.s.} \quad \forall \xi \in \mathrm{L}^0(\mathcal{F}_t, \mathbb{P})$$

◊

Theorem 2.5.3. *(NA) holds if and only if $(NA)_t$ holds for every $t \in \mathbb{T} \setminus \{T\}$*

Proof. "$\Rightarrow$": Let us make counter-assumption that there exists such $\xi \in \mathrm{L}^0(\mathcal{F}_t, \mathbb{P})$ that $(\mathrm{NA})_t$ does not hold for some $t \in \mathbb{T} \setminus \{T\}$. Now strategy $\phi_t = \xi$ and $\phi_s = 0$ for $s \in \mathbb{T} \setminus \{t\}$ violates the (NA) condition.
"$\Leftarrow$": Let us assume that (NA) does not hold, that is there exists self-financing predictable strategy ϕ such that $V_0^\phi = 0$, $V_T^\phi \geq 0$ ($\mathbb{P}$-a.s.) and $V_T^\phi > 0$ with positive probability. We define

$$u = \inf\{t \in \mathbb{T} \setminus \{T\} : \mathbb{P}(V_{t+1}^\phi \geq 0) = 1,\ \mathbb{P}(V_{t+1}^\phi > 0) > 0\}.$$

Recall that the value of self-financing portfolio increases only via increments in asset prices, that is

$$\Delta V_{u+1}^\phi = V_{u+1}^\phi - V_u^\phi = \phi_{u+1} \cdot \Delta S_{u+1}.$$

Now let us first consider the case, when $V_u^\phi \leq 0$ ($\mathbb{P}$-a.s.) which follows that $\phi_{u+1} \cdot \Delta S_{u+1} \geq 0$ ($\mathbb{P}$-a.s.) and strictly positive with positive probability. We can choose $\xi = \phi_{u+1}$ and have immediately that $\xi \cdot \Delta \widetilde{S}_{t+1} \geq 0$ ($\mathbb{P}$-a.s.) and strictly positive with positive probability. So the $(\mathrm{NA})_t$ condition is not satisfied in this case at time $t = u$.

Then let us consider the remaining case, when $V_u^\phi > 0$ with positive probability. From the way we defined u, we get that also $V_u^\phi < 0$ with positive probability. In this case we choose $\xi = \phi_{u+1} \mathbb{1}_{\{V_u^\phi < 0\}}$, which yields

$$\xi \cdot \Delta \widetilde{S}_{u+1} = (\phi_{u+1} \cdot \Delta \widetilde{S}_{u+1}) \mathbb{1}_{\{V_u^\phi < 0\}} = (\widetilde{V}_{u+1}^\phi - \widetilde{V}_u^\phi) \mathbb{1}_{\{V_u^\phi < 0\}} \geq 0 \quad \mathbb{P}\text{-a.s.}$$

and strictly positive with positive probability, since $\mathbb{P}(V_u^\phi < 0) > 0$. Again we have been able to create an arbitrage at time $t = u$. Thus the proof is complete. $\square$

2.6 Geometric interpretation of arbitrage

In this section we restrict ourselves to one-period and study arbitrage-free prices from the geometric perspective. The following notation is used throughout the section. We assume that $S = (S^1, \ldots, S^d)$ is the vector of final prices of risky assets with initial prices $\pi = (\pi^1, \ldots \pi^d)$. For non-risky numeraire asset the final price is denoted by S^0 and initial price π^0. Other notations remain the same.

Let us introduce Borel probability measure μ on $\mathbb{R}^d$ such that

$$\mu(B) = \mathbb{P}(\widetilde{S} \in B) \text{ for all Borel sets } B \subset \mathbb{R}^d$$

Definition 2.6.1. Let $A \subset \mathbb{R}^d$ be the smallest closed set such that $\mu(A^c) = 0$. We call set A the *support* of μ and denote it by $\mathrm{supp}(\mu)$. $\diamondsuit$

For every Borel probability measure on $\mathbb{R}^d$ unique support exists (see Proposition 1.45 in [10]). We say that set C in an underlying vector space is *convex* if

$$x, y \in C \text{ and } \lambda \in [0,1] \ \Rightarrow\ (\lambda x + (1-\lambda) y) \in C.$$

The smallest convex set containing $A \subset \mathbb{R}^d$ is called the *convex hull* of A. Next definition will give us useful equivalent characterization of the convex hull.

Definition 2.6.2. The *convex hull* of nonempty set $A \subset \mathbb{R}^d$ is denoted by $\operatorname{conv}(A)$ and defined by

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \,\middle|\, \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1,\ x_i \in A,\ n \in \mathbb{N} \right\}$$

$\Diamond$

In the geometric characterization of arbitrage-free prices, we are only interested in prices that belong "strictly" to $\operatorname{conv}(\operatorname{supp}(\mu))$ and thus guarantee the existence of an equivalent martingale measure. So we need the next definition.

Definition 2.6.3. The *relative interior* of a convex set $C \subset \mathbb{R}^d$ denoted by $\operatorname{ri}(C)$ is the set of all points $x \in C$ such that for all $y \in C$ there exists $\varepsilon > 0$ with $(x - \varepsilon(y - x)) \in C$ $\Diamond$

The next theorem is the main result of this section. We are only interested in applications of the final result and thus will not motivate this result. Reader is refered to Section 1.4 of [10] for details.

Theorem 2.6.4. *Let μ be the distribution of the vector of discounted final prices $\widetilde{S}$ of risky assets. Then one-period market model is arbitrage-free if and only if the vector of initial prices π belongs to* $\operatorname{ri}(\operatorname{conv}(\operatorname{supp}(\mu)))$*, that is the relative interior of the convex hull of the support of μ.*

Example 2.6.5. Suppose random variable X is binomially distributed, that is

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

with parameter $p = \frac{1}{2}$ and $n = 4$. Let B be the non-risky numeraire asset with constant final and initial price $B = \pi^B = 1$ (for example a bank account with zero interest rate) and risky asset (for example stock) $S = \frac{1}{2}X$ with initial price π^S. We introduce European call option (see Section 2.7 for details) $C = (S - K)^+$, with strike price $K = 1$ and initial price π^C. Let us set $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3, \omega_4\}$ and define $\{\omega_k\} = \{\omega \in \Omega \,|\, X(\omega) = k\} = \{X = k\}$ thus

$$\mathbb{P}(\{\omega_k\}) = \mathbb{P}(X = k) \text{ for } k \in \{0, 1, 2, 3, 4\}.$$

In Table 2.1 we have presented realizations for the asset prices with respect to different scenarios. We define

$$\mu(\{(\pi^S, \pi^C)\}) = \mathbb{P}((\widetilde{S}, \widetilde{C}) = (\pi^S, \pi^C))$$

and notice that

$$\operatorname{supp}(\mu) = \{(0,0), (1/2, 0), (1, 0), (3/2, 1/2), (2, 1)\},$$

since these are the only pairs with positive probability. The support of μ is drawn in Figure 2.2 as thick points and the convex hull of the support is the triangle generated by those points including the boundary presented by dashed lines. Now by Theorem 2.6.4 we know that the pairs of arbitrage-free prices $(\pi^S, \pi^C) \in \mathbb{R}^2$ belong to coloured area seen

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$\mathbb{P}(\{\omega_k\})$	1/16	1/4	3/8	1/4	1/16
$X(\omega_k)$	0	1	2	3	4
$\widetilde{B}(\omega_k)$	1	1	1	1	1
$\widetilde{S}(\omega_k)$	0	1/2	1	3/2	2
$\widetilde{C}(\omega_k)$	0	0	0	1/2	1

Table 2.1: Realizations of final asset prices.

in Figure 2.2 excluding the boundary points shown as dashed lines. Explicitly the area is determined by 3 lines: the upper boundary $\pi^C = \frac{1}{2}\pi^S$ and lower boundaries $\pi^C = \pi^S - 1$ and $\pi^C = 0$ so the area is

$$\begin{aligned}&\{(\pi^S, \pi^C) \in \mathbb{R}^2 \,|\, \pi^C > 0,\ \pi^C > \pi^S - 1,\ \pi^C < \frac{1}{2}\pi^S\}\\ &= \{(\pi^S, \pi^C) \in \mathbb{R}^2 \,|\, (\pi^S - 1)^+ < \pi^C < \frac{1}{2}\pi^S\}\end{aligned}$$

For example, if we fix the stock price $\pi^S = 3/2$, then we see that the option price π^C belongs to the open interval $(\frac{1}{2}, \frac{3}{4})$, when the market is arbitrage-free.

Figure 2.2 is also useful for finding out the super- or sub-hedging strategy (see Definition 2.8.1) for the option. The hypotenuse of the triangle generated by dashed lines gives guideline how to determine the cheapest superhedge. Let us define portfolio consisting of η amount of money in bank account B and ψ amount of stock S. The value of superhedging portfolio must satisfy

$$\eta B + \psi S(\omega) \geq C(\omega) \text{ for all } \omega \in \Omega. \tag{2.6.6}$$

Now choosing the portfolio suggested by the hypotenuse, we get $\eta = 0$ and $\psi = 1/2$. This is a superhedging strategy since $\psi S(\omega)$ is clearly larger than (or equal to) $C(\omega)$ for $\omega \in \{\omega_0, \omega_1, \omega_2\}$ and also for $\omega \in \{\omega_3, \omega_4\}$ it meets the demands. From the figure we see that there is no cheaper superhedging strategy available. △

2.7 European options

Definition 2.7.1. A *European derivative* with underlying assets $S = (S^1, \dots, S^d)$ is a random variable $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ which is measurable with respect to $\mathcal{F}_T^S = \sigma\{S_t \mid t \leq T\}$. We denote $X_T := X$ for the payoff (or the claim) of the derivative at time T. ◊

In this thesis we are particularly interested in the one type of derivatives, options. An *option* on an asset is a contract giving the owner the right, but not the obligation, to trade the asset for a fixed price at a future date. We call last date on which option can

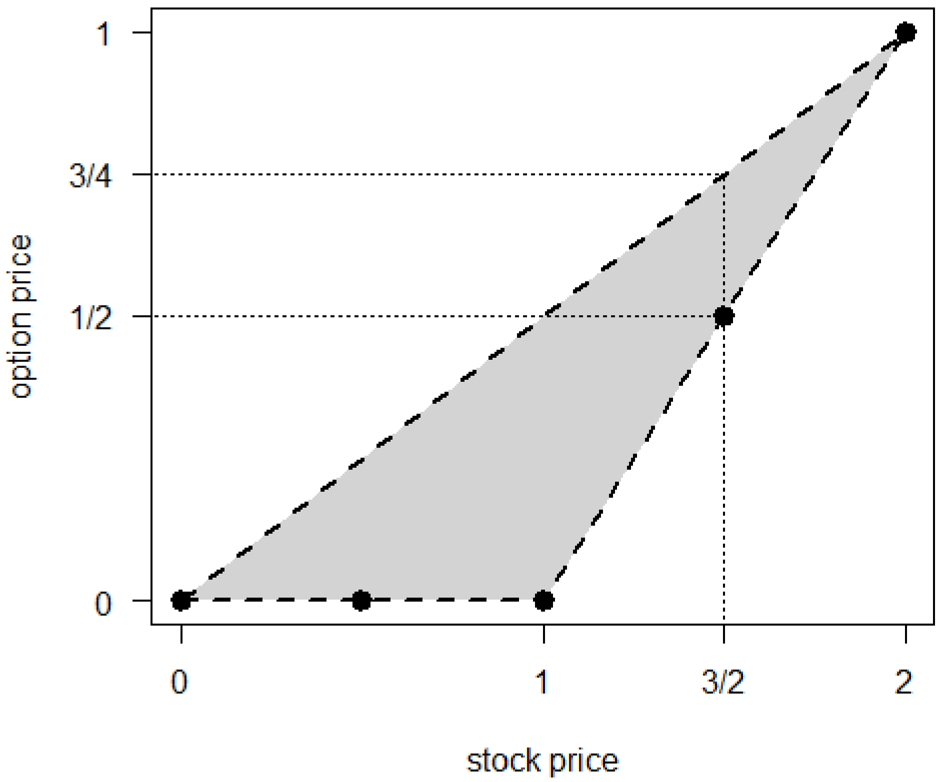

Figure 2.2: Geometric characterization of arbitrage-free prices

be *exercised* (according to its terms) the *expiration date* T. The option is *European*, if it can only be exercised at the fixed expiration date T. Let us denote S^i for a price process of a underlying risky asset of an option in this section.

A *European call option* gives the buyer (of the option) a right, but not an obligation, to buy the asset whose real price is S^i_T at time T at the *strike price* K. If the real price of the asset (at time T) is higher than the strike price, then the buyer of the option can buy the asset at price K and immediately sell it at price S^i_T. Thus the buyer gets the difference $S^i_T - K$ as a payoff. If the real price of the asset is lower than the strike price, then the option expires worthless. Therefore we can write the payoff of a European call option as

$$C_T = (S^i_T - K)^+,$$

where we used notation $(\cdot)^+ = \max\{\,\cdot\,, 0\}$.

A *European put option* gives the buyer (of the option) a right, but not an obligation, to sell the asset whose real price is S^i_T at time T at the strike price K. If the real price of the asset (at time T) is lower than the strike price, then the buyer of the option can buy the asset at price S^i_T and immediately sell it at price K. Thus the buyer gets the difference $K - S^i_T$ as a payoff. If the real price of the asset is higher than the strike price, then the option expires worthless. Therefore we can write the payoff of a European put option as

$$P_T = (K - S^i_T)^+.$$

Consider the relationship between the payoffs of the call and the put option. We see that the following equality holds

$$C_T - P_T = (S^i_T - K)^+ - (K - S^i_T)^+ = S^i_T - K,$$

regardless of which one of the prices (real or strike) is higher. Let us further denote c^C_t and c^P_t for the price of the call option and the put option at time $t \in \mathbb{T}$, respectively. If we assume that market is arbitrage-free, then there exists equivalent martingale measure $\mathbb{Q}$ (with numeraire S^0) so that

$$c^C_t = S^0_t \mathbb{E}_{\mathbb{Q}}(\widetilde{C}_T | \mathcal{F}_t) \quad \text{and} \quad c^P_t = S^0_t \mathbb{E}_{\mathbb{Q}}(\widetilde{P}_T | \mathcal{F}_t)$$

are the arbitrage-free time t prices of those options (see the next section for details). Hence evidently we have the following result:

Proposition 2.7.2. *(Put-Call parity) In an arbitrage-free market model*

$$c^C_t - c^P_t = S^i_t - \frac{S^0_t}{S^0_T} K, \tag{2.7.3}$$

for every $t \in \mathbb{T}$.

Previous proposition enables us to concentrate our attention on call options alone, since we can always (in an arbitrage-free market) calculate the price of a put option by means of the price of a call option.

In general, derivative X is called *path-independent* if the payoff of X only depends on the time T values of assets. That is, if the payoff of X can be represented in the form $X_T = f(S_T)$ for some function f. For example, a European call option falls into this category.

On the contrary, *path-depend* derivative X has a payoff that can depend also on the earlier values of assets. For example, the payoff of an *Asian call option* with strike price K can depend on the average price asset i. Thus it is path-depend with the payoff

$$X_T = \left(\frac{1}{T} \sum_{t=0}^{T} S_t^i - K \right)^+ . \tag{2.7.4}$$

Loosely speaking, European and American (type which is introduced later) call and put options are sometimes called *vanilla* options. More complex, often path-dependent, options are called *exotic* options. There exist a vast number of different types of options for all kinds of purposes.

2.8 Pricing and hedging

Definition 2.8.1. We say that derivative X is *replicable* (or *hedgeable*), if there exists predictable self-financing strategy θ that takes at expiration date T the same value of a derivative X, that is

$$V_T^\theta = X_T \tag{2.8.2}$$

almost surely. We call such θ a *replicating strategy* for X. In the cases of $V_T^\theta \geq X_T$ and $V_T^\theta \leq X_T$ we call θ a *super*- and *sub*-replicating strategy, respectively. $\Diamond$

Example 2.8.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex and smooth function and $X_T = f(S_T^i)$ derivative with underlying asset S^i

$$\begin{aligned}
X_T = f(S_T^i) &= f(0) + \int_0^{S_T^i} f'(x)dx \\
&= f(0) + \int_0^{S_T^i} \left(f'(0) + \int_0^x f''(k)dk \right) dx \\
&= f(0) + f'(0)S_T^i + \int_0^{S_T^i} \left(\int_0^\infty \mathbb{1}_{\{k<x\}} f''(k)dk \right) dx \\
&= f(0) + f'(0)S_T^i + \int_0^\infty \left(\int_0^{S_T^i} \mathbb{1}_{\{k<x\}} dx \right) f''(k)dk \\
&= f(0) + f'(0)S_T^i + \int_0^\infty (S_T^i - k)^+ f''(k)dk
\end{aligned}$$

We notice that X can be replicated by depositing $f(0)$ amount to a bank account (or borrowing from the bank in case of negative value), purchasing $f'(0)$ amount of asset (or short-selling in case of negative value) and trading European call options with different strike prices. In practice, though, there are only finitely many strike prices available for options in the market so the integral must be approximated with a sum.

Moreover we can relax the smoothness condition, but then we have to consider left and right differential coefficients and different measure for the integral. △

Recall that we denote the set of equivalent martingale measures by

$$\mathcal{M}^b = \Big\{ \mathbb{Q} \sim \mathbb{P} \mid \widetilde{S} \text{ is martingale under } \mathbb{Q} \text{ and } d\mathbb{Q}/d\mathbb{P} \in L^\infty(\mathbb{P}) \Big\}.$$

Set $\mathcal{M}^b$ is convex, meaning that if $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}^b$ and $\lambda \in [0,1]$, then also

$$\lambda \mathbb{Q}_1 + (1-\lambda)\mathbb{Q}_2 \in \mathcal{M}^b.$$

We define the set of arbitrage-free initial prices of discounted derivative $\widetilde{X}$ by

$$\Pi(\widetilde{X}) = \Big\{ \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T) \mid \mathbb{Q} \in \mathcal{M}^b \text{ and } \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T) < \infty \Big\}$$

and the lower and upper boundaries of $\Pi(\widetilde{X})$ by

$$\pi^{\widetilde{X}}_{\text{inf}} = \inf_{\mathbb{Q} \in \mathcal{M}^b} \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T) \quad \text{and} \quad \pi^{\widetilde{X}}_{\text{sup}} = \sup_{\mathbb{Q} \in \mathcal{M}^b} \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T)$$

respectively. From the convexity of $\mathcal{M}^b$ and the linearity of expectation, we deduce that $\Pi(\widetilde{X})$ is an interval and if $\mathcal{M}^b$ is a singleton (a set with exactly one element), then $\Pi(\widetilde{X}$ must be also singleton (only one measure uniquely defines expectation). We will later see further that if $\mathcal{M}^b = \{\mathbb{Q}\}$, then all derivatives are replicable (and that this also holds on the reversed direction). Let us compile previous observations with some extra conditions:

Proposition 2.8.4. *Let $\widetilde{X}$ be discounted derivative, then either*

(i) $\widetilde{X}$ is replicable and the set of arbitrage-free initial prices is singleton

$$\Pi(\widetilde{X}) = \{\pi^{\widetilde{X}}\}, \quad \textit{and} \quad \pi^{\widetilde{X}} \geq 0.$$

(ii) $\widetilde{X}$ is not replicable and the set of arbitrage-free initial prices is open interval

$$\Pi(\widetilde{X}) = (\pi^{\widetilde{X}}_{\text{inf}}\,, \pi^{\widetilde{X}}_{\text{sup}}), \quad \textit{and} \quad \pi^{\widetilde{X}}_{\text{inf}} < \pi^{\widetilde{X}}_{\text{sup}}.$$

Proof. See the proof of Theorem 5.33 in [10] for openness of $\Pi(\widetilde{X})$ and some technical details. □

Example 2.8.5. Consider the market model described in Example 2.6.5. Let us try to replicate option C with a portfolio consisting of η amount of money in bank account B and ψ amount of stock S. The final value of such portfolio must coincide with the final value C. So in order to replicate

$$\eta B + \psi S(\omega) = C(\omega) \tag{2.8.6}$$

has to hold for all $\omega \in \Omega$. We can use the table of asset prices from aforementioned example, thus equation (2.8.6) yields

$$\begin{cases} \eta = 0, & \omega = \omega_0 \\ \eta + \psi = 0, & \omega = \omega_2 \\ \eta + 2\psi = 1, & \omega = \omega_4. \end{cases}$$

From the first two equation, we get that $\eta = \psi = 0$ which contradicts the last equation. Hence C is not replicable in this model (and clearly $\widetilde{C}$ is not either). Notice that discounting in this model does not change values since $B = \pi^B = 1$.

Let us fix the initial price of stock $\pi^S = 3/2$ and calculate the set of arbitrage-free initial prices for the discounted option. We use the conditions of equivalent martingale measure $\mathbb{Q}$ (with $q_i := \mathbb{Q}(\{\omega_i\}) \in (0,1)$, for $i = 0, 1, 2, 3, 4$) to find bounds for π^C:

$$\begin{cases} \mathbb{E}_{\mathbb{Q}}(\widetilde{S}) = \pi^S & \iff \quad \frac{1}{2}q_1 + q_2 + \frac{3}{2}q_3 + 2q_4 = \frac{3}{2} \qquad (*) \\ \mathbb{E}_{\mathbb{Q}}(\widetilde{C}) = \pi^C & \iff \quad \frac{1}{2}q_3 + q_4 = \pi^C. \qquad (\dagger) \end{cases}$$

From $(*)$ we deduce $q_4 < \frac{3}{4} - \frac{3}{4}q_3$ which together with $(\dagger)$ yields the upper bound:

$$\pi^C = \frac{1}{2}q_3 + q_4 < \frac{3}{4} - \frac{1}{4}q_3 < \frac{3}{4}.$$

Since $\mathbb{Q}$ is a probability measure, we must have $\sum_{i=0}^4 q_i = 1$ and hence $-q_2 > q_1+q_3+q_4-1$, so $(*)$ gives

$$2q_4 = \frac{3}{2} - \frac{1}{2}q_1 - \frac{3}{2}q_3 - q_2 > \frac{1}{2} + \frac{1}{2}q_1 - \frac{1}{2}q_3 + q_4 > \frac{1}{2} - \frac{1}{2}q_3 + q_4.$$

That is $q_4 > \frac{1}{2} - \frac{1}{2}q_3$, so again combined with $(\dagger)$ we get the lower bound:

$$\pi^C = \frac{1}{2}q_3 + q_4 > \frac{1}{2}.$$

We have observed that non-replicable discounted option $\widetilde{C}$ has arbitrage-free initial prices that belong to open interval $\Pi(\widetilde{C}) = (\frac{1}{2}, \frac{3}{4})$. This coincides with Proposition 2.8.4 and the result we got in Example 2.6.5 by geometrically studying the arbitrage-free prices. Although we did not explicitly write down the different measures, we implicitly used two different measures to determine bounds for the interval, when we gave estimates for parameters q_i to get inequations above. $\triangle$

Recall that Corollary 2.4.18 guarantees that, if the final portfolio values coincide in the arbitrage-free market, then the values have to coincide at any earlier moment of time. This statement is sometimes called the *principle of no arbitrage*. This means that it suffices to find the initial value of a replicating strategy for a derivative in order to price the derivative. Let us denote the family of predictable and self-financing strategies by Θ and the families of super- and sub-replicating strategies for the derivative X by

$$\Theta_X^+ = \{\theta \in \Theta \mid V_T^\theta \geq X_T\} \quad \text{and} \quad \Theta_X^- = \{\theta \in \Theta \mid V_T^\theta \leq X_T\}.$$

Lemma 2.8.7. *For every equivalent martingale measure $\mathbb{Q}$, we have*

$$\sup_{\theta\in\Theta_X^-} \widetilde{V}_t^\theta \leq \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T|\mathcal{F}_t) \leq \inf_{\theta\in\Theta_X^+} \widetilde{V}_t^\theta,$$

for every $t \in \mathbb{T}$.

Proof. Let $\theta \in \Theta_X^+$ so that $X_T \leq V_T^\theta$ for every θ. Thus, by Proposition 2.4.16, we have

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T|\mathcal{F}_t) \leq \mathbb{E}_{\mathbb{Q}}(\widetilde{V}_T^\theta|\mathcal{F}_t) = \widetilde{V}_t^\theta,$$

and in a similar fashion we can prove the reversed inequality for $\theta \in \Theta_X^-$. □

For a fixed equivalent martingale measure $\mathbb{Q}$ with numeraire S^0, let us define a process by setting

$$c_t^X(\mathbb{Q}) := S_t^0\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T|\mathcal{F}_t). \tag{2.8.8}$$

We say that process $c^X(\mathbb{Q}) = (c_t^X(\mathbb{Q}))_{t\in\mathbb{T}}$ is the $\mathbb{Q}$-*risk neutral price* of X. Notice that it indeed depends on the chosen equivalent martingale measure $\mathbb{Q}$. In the following result we see that, if X is replicable, then every $\mathbb{Q}$ agree with the same price and thus the price is independent of $\mathbb{Q}$ and unique. This is the basis of arbitrage-free pricing.

Theorem 2.8.9. *If X is a replicable derivative in an arbitrage-free market, then for every replicating strategy θ and for every equivalent martingale measure $\mathbb{Q}$, we have*

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T|\mathcal{F}_t) = \widetilde{V}_t^\theta, \quad t \in \mathbb{T}. \tag{2.8.10}$$

We call the process $(c_t^X)_{t\in\mathbb{T}}$ defined by $c_t^X := V_t^\theta$ the arbitrage-free price of X.

Proof. Let θ and ϕ be replicating strategies for X. Since the market is arbitrage-free and time T values of θ, ϕ, and X coincide, according to Corollary 2.4.18 also all earlier values match one another. Furthermore, because $\theta \in \Theta_X^+ \cap \Theta_X^-$, Lemma 2.8.7 yields

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T|\mathcal{F}_t) = \widetilde{V}_t^\theta$$

for every martingale measure $\mathbb{Q}$. □

Example 2.8.11. Consider model defined in Example 2.2.4. We add a new asset, European call option $C_2 = (S_2 - K)^+$ with strike price $K = 140$, to the model. We know stock values S_2 at time $t = 2$ so we can straightaway calculate the payoff of the option:

$$C_2(\omega) = \begin{cases} 154, & \omega = (1,1) \\ 28, & \omega \in \{(1,0),(0,1)\} \\ 0, & \omega = (0,0) \end{cases}$$

Let us find replicating strategy $\theta = \{\theta_t \,|\, t = 1,2\} = \{(\eta_t, \psi_t) \,|\, t = 1,2\}$ of C_2 using bank account B and stock S for. The value of strategy θ has to equal the payoff of the option at time $t = 2$, that is

$$V_2^\theta(\omega_1,\omega_2) = \eta_2(\omega_1)B_2 + \psi_2(\omega_1)S_2(\omega_1,\omega_2) = C_2(\omega_1,\omega_2) \quad \forall\omega = (\omega_1,\omega_2) \in \Omega, \tag{2.8.12}$$

where we denoted $\eta_2(\omega_1) := \eta_2(\omega_1, \omega_2)$ and $\psi_2(\omega_1) := \psi_2(\omega_1, \omega_2)$ since θ is predictable and thus η_2 and ψ_2 do not depend on ω_2. Now equation (2.8.12) gives us four condition for η_2 and ψ_2, with different $\omega_1, \omega_2 \in \{0, 1\}$. Let us first fix $\omega_1 = 1$, that yields

$$\begin{cases} \eta_2(1) \cdot 1.21 + \psi_2(1) \cdot 294 = 154, & \text{when } \omega_2 = 1 \\ \eta_2(1) \cdot 1.21 + \psi_2(1) \cdot 168 = 28, & \text{when } \omega_2 = 0, \end{cases}$$

from which we can solve $\eta_2(1) \approx -115.7$ and $\psi_2(1) = 1$. In a similar fashion we fix $\omega_1 = 0$ and get corresponding two equations from which we solve $\eta_2(0) \approx -30.9$ and $\psi_2(0) \approx 0.4$.

Recall that replicating strategy must be self-financing, that is

$$\eta_1 B_1 + \psi_1 S_1(\omega_1) = \eta_2(\omega_1) B_1 + \psi_2(\omega_1) S_1(\omega_1), \text{ for } \omega_1 = 0, 1,$$

where η_1 and ψ_1 are deterministic. From the self-financing condition, we can solve $\eta_1 \approx -73.3$ and $\psi_1 \approx 0.8$.

So we can replicate C_2 by first, at time $t = 0$, investing $\eta_1 \approx -73.3$ in the bank account (borrowing money) and buying $\psi_1 \approx 0.8$ amount of stock. Then, at time $t = 1$, we choose corresponding amounts

$$\eta_2(\omega_1) \approx \begin{cases} -30.9, & \omega_1 = 0 \\ -115.7, & \omega_1 = 1, \end{cases} \quad \text{and} \quad \psi_2(\omega_1) \approx \begin{cases} 0.4, & \omega_1 = 0 \\ 1, & \omega_1 = 1, \end{cases}$$

depending on whether the stock price has gone up ($\omega_1 = 1$) or down ($\omega_1 = 0$). Now we can use theorem 2.8.9 and replicating strategy to determine the arbitrage-free price of C_2. According to aforementioned theorem, the value of the replicating strategy at times $t = 0, 1$ is the arbitrage-free price of the option at those times. At time $t = 0$, the price is

$$c_0^{C_2} = V_0^\theta = \eta_1 B_0 + \psi_1 S_0 \approx 43.4.$$

This is also the price of the hedging for the option. At time $t = 1$, we calculate the price similarly

$$c_1^{C_2}(\omega_1) = V_1^\theta = \eta_1 B_1 + \psi_1 S_1(\omega_1) \approx \begin{cases} 82.7, & \omega_1 = 1 \\ 12.7, & \omega_1 = 0, \end{cases}$$

which naturally also depends on whether the stock price has gone up or down. △

Often derivative X is not replicable so there is no replicating strategy for X and hence Theorem 2.8.9 cannot be used. Fortunately, formula (2.8.8) provides way to price X so that the market remains arbitrage-free, although the price is not unique since it depends on $\mathbb{Q}$. The next corollary predicates that the aforementioned formula does not yield unique price for non-replicable X.

Corollary 2.8.13. *In an arbitrage-free market, derivative X is replicable if and only if $\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_T)$ has the same value for every equivalent martingale measure $\mathbb{Q}$.*

Proof. "⇒": Follows directly from Theorem 2.8.9.
"⇐": Follows directly from Proposition 2.8.4. □

As we already discussed earlier, we have:

Proposition 2.8.14. *For any equivalent martingale measure $\mathbb{Q}$, the market model with riskless asset S^0 and risky assets $S^1, \dots, S^d$ and the price process $c^X(\mathbb{Q})$ (defined by (2.8.8)) is arbitrage-free.*

Proof. Since $c^X(\mathbb{Q})$ is a $\mathbb{Q}$-martingale (by its definition) then the augmented market with $c^X(\mathbb{Q})$ has the same equivalent martingale measure $\mathbb{Q}$. Hence by the first fundamental theorem of asset pricing (Theorem 2.4.15) the augmented market is arbitrage-free. □

2.9 Market completeness

Recall from the previous section, that a replicable (European) derivative has unique arbitrage-free price process determined by the equivalent martingale measure. In some market models, it is possible to replicate every (European) derivative and thus finding right prices is particularly convenient.

Definition 2.9.1. We say that a market model is *complete* if every European derivative is replicable in it. On the contrary, if there exist a European derivative that can not be replicated, we say that the market model is *incomplete*. ♢

Let us for a moment assume that we have an arbitrage-free and complete market model. From the first fundamental theorem, we have that there exist at least one equivalent martingale measure. Let us assume that $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are such probability measures in our model. By the completeness of model, derivative $X = \mathbb{1}_A$ with $A \in \mathcal{F}$ is replicable. Thus by Corollary 2.8.13, we have

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\widehat{\mathbb{Q}}}(X) \iff \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A) = \mathbb{E}_{\widehat{\mathbb{Q}}}(\mathbb{1}_A) \iff \mathbb{Q}(A) = \widehat{\mathbb{Q}}(A) \tag{2.9.2}$$

for every $A \in \mathcal{F}$. So the completeness (together with arbitrage-freeness) implies uniqueness of equivalent martingale measure. The same corollary (2.8.13) also implies the converse. That is, if there exists only one equivalent martingale measure in model, then the expectation is always unique for each derivative and hence every (European) derivative is replicable.

Theorem 2.9.3. *(Second fundamental theorem of asset pricing) An arbitrage-free discrete-time market model is complete if and only if there exist a* ***unique*** *equivalent martingale measure.*

Proof. Discussion above motivates the result with a small amount of details, since it relies on the mentioned corollary. For more details, see for example, the proof of Theorem 4.1.2 in [9]. □

The completeness of a market model requires a certain structure from the underlying probability space. We will next study this required structure.

Definition 2.9.4. Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $A \subset \Omega$ in $\mathcal{F}$ is called an *atom* if $\mathbb{P}(A) > 0$ and for any subset $B \subset A$ in $\mathcal{F}$ with $\mathbb{P}(B) < \mathbb{P}(A)$ it holds that $\mathbb{P}(B) = 0$. ♢

A complete market model must have such underlying probability space, that Ω can be separated into finitely many atoms. Also the the number of atoms can be $(d+1)^T$ at most, where $d \in \mathbb{N}$ is the number of risky asset. We will not prove this requirement. See, for example, the ending of the proof of Theorem 5.37 in [10].

Another property of an arbitrage-free and complete market tells that every martingale (under $\mathbb{Q}$) can be represented in terms of discounted asset price processes. This property can be extended to continuous-time models by replacing martingale transforms with stochastic integrals.

Proposition 2.9.5. *(Martingale representation property) An arbitrage-free market model with equivalent martingale measure $\mathbb{Q}$ is complete if and only if each $(\mathbb{F}, \mathbb{Q})$-martingale $M = \{M_t \mid t \in \mathbb{T}\}$ can be represented in the form*

$$M_t = M_0 + \sum_{u=1}^{t} \phi_u \cdot \Delta \widetilde{S}_u = M_0 + \sum_{i=1}^{d} (\phi^i \bullet \widetilde{S}^i)_t, \tag{2.9.6}$$

where ϕ is some predictable process.

Proof. See, for example, the proof of Theorem 5.38 in [10] or Proposition 4.2.1 in [9]. □

2.10 Hedging in incomplete market

This section depends highly on the properties of the underlying L^2 Hilbert space which was introduced in the preliminary section. We review different ways to hedge derivatives, when the full replication with a self-financing strategy is not possible. We no longer restrict ourselves to self-financing portfolios. Let us consider trading strategy $\xi = \{(\xi_t^0, \xi_t^1, \dots, \xi_t^d) \mid t \in \mathbb{T}\}$, where ξ^0 is adapted and ξ^i is predictable process for all $i = 1, 2, \dots, d$. We assume that value of such portfolio is given by

$$V_0^\xi = \xi_0^0 \quad \text{and} \quad V_t^\xi = \xi_t^0 S_t^0 + \xi_t^* \cdot S_t \quad \forall t \in \mathbb{T} \setminus \{0\},$$

where S^0 denotes price process of riskless asset and S denotes the price process of risky assets contrary to earlier sections and ξ^* tags portfolio of risky assets. For simplicity and instead of discounting we set $S_t^0 = 1$ for all $t \in \mathbb{T}$. We will carry out this convention throughout the section. The gain process of strategy ξ is now given by

$$G_0^\xi = 0 \quad \text{and} \quad G_t^\xi = \sum_{u=1}^{t} \xi_u^* \cdot \Delta S_u \quad \forall t \in \mathbb{T} \setminus \{0\},$$

where $\Delta S_u = S_u - S_{u-1}$. We want to keep up with cumulated cost of the hedging strategy, so we introduce the *cost process* of strategy ξ denoted by C^ξ and defined by

$$C_t^\xi = V_t^\xi - G_t^\xi, \quad \forall t \in \mathbb{T}.$$

Remark 2.10.1. Earlier we considered only self-financing strategies for which the cost process would have been constant (the initial value of the portfolio). △

We assume that X is a European derivative for which holds $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Likewise we demand that $S_t \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in \mathbb{T}$. Furthermore, we say that strategy ξ is L^2-*admissible strategy* for X if

$$V_T^{\xi} = X \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad V_t^{\xi} \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}) \quad \forall t \in \mathbb{T}.$$

We want to find L^2-admissible strategy $\widehat{\xi}$ such that for all $t \in \mathbb{T} \setminus \{T\}$

$$\mathbb{E}[(\Delta C_{t+1}^{\widehat{\xi}})^2 | \mathcal{F}_t] \leq \mathbb{E}[(\Delta C_{t+1}^{\xi})^2 | \mathcal{F}_t] \quad \mathbb{P}\text{-a.s.} \tag{2.10.2}$$

for each L^2-admissible strategy ξ. Such strategy $\widehat{\xi}$ is called ***locally risk-minimizing***. As before, we use Δ to represent change of value of the process in question at given time, that is, $\Delta C_{t+1}^{\widehat{\xi}} = C_{t+1}^{\widehat{\xi}} - C_t^{\widehat{\xi}}$. Using the expression (2.10.12), we can equivalently solve our minimizing problem by finding minimum for

$$\mathbb{E}\big[\big(V_{t+1}^{\xi} - (V_t^{\xi} + \xi_{t+1}^{*} \cdot \Delta S_{t+1})\big)^2 | \mathcal{F}_t\big]. \tag{2.10.3}$$

This minimizing is related to linear least squares method for estimating the unknown parameters (in this particular case these would be V_t^{ξ} and ξ_{t+1}^{*}) in a linear regression model, provided that we knew value V_{t+1}^{ξ}. Next definition is a substitute for self-financing condition that was used particularly in complete markets.

Definition 2.10.4. Strategy ξ is called *mean self-financing* if it is L^2-admissible and

$$\mathbb{E}(\Delta C_{t+1}^{\xi} | \mathcal{F}_t) = 0, \quad \mathbb{P}\text{-a.s.}$$

for all $t \in \mathbb{T} \setminus \{T\}$. $\Diamond$

We notice that mean self-financing strategy has cost process with martingale property. Recall that martingale constant on average, so on average the cost of hedging will be the initial value of the portfolio. We will need a few definition before stating the main result. Let us assume that $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) =: L^2(\mathbb{P})$, then *conditional covariance* of X and Y with respect to sigma-algebra $\mathcal{G} \subset \mathcal{F}$ is denoted and defined by

$$\mathrm{Cov}(X, Y | \mathcal{G}) := \mathbb{E}(XY | \mathcal{G}) - \mathbb{E}(X | \mathcal{G}) \mathbb{E}(Y | \mathcal{G}) \tag{2.10.5}$$

Similarly *conditional variance* of $X \in L^2(\mathbb{P})$ is denoted and defined by

$$\mathrm{Var}(X | \mathcal{G}) := \mathbb{E}(X^2 | \mathcal{G}) - (\mathbb{E}(X | \mathcal{G}))^2. \tag{2.10.6}$$

In multidimensional case, where $X = (X_1, \ldots, X_d)$ and $Y, X_1, \ldots, X_d \in L^2(\mathbb{P})$, we have

$$\mathrm{Cov}(X, Y | \mathcal{G}) := [\mathrm{Cov}(X_1, Y | \mathcal{G}), \ldots, \mathrm{Cov}(X_d, Y | \mathcal{G})]. \tag{2.10.7}$$

Also for multidimensional X we have that $\mathrm{Cov}(X | \mathcal{G})$ is $(d \times d)$-matrix with elements $\mathrm{Cov}(X_i, X_j | \mathcal{G})$ for $i, j = 1, \ldots, d$.

Definition 2.10.8. Two $\mathbb{F}$-adapted processes $X = \{X_t \,|\, t \in \mathbb{T}\}$ and $Y = \{Y_t \,|\, t \in \mathbb{T}\}$ are called *strongly orthogonal* to each other under $\mathbb{P}$ if

$$\mathbb{E}(\Delta X_{t+1} \Delta Y_{t+1} | \mathcal{F}_t) = 0, \quad \mathbb{P}\text{-a.s.} \tag{2.10.9}$$

for every $t \in \mathbb{T} \setminus \{T\}$. $\diamondsuit$

Strongly orthogonal condition resembles orthogonality in L^2 with respect to inner product $\langle X, Y \rangle = \mathbb{E}(XY)$ with $X, Y \in L^2$. Hence the term strongly orthogonal. We are finally ready to state the main result for locally risk-minimizing strategies.

Theorem 2.10.10. *Let us assume that $\widehat{\xi}$ is L^2-admissible strategy. Then $\widehat{\xi}$ is locally risk-minimizing if and only if $\widehat{\xi}$ is mean self-financing and its cost process $C^{\widehat{\xi}}$ is strongly orthogonal to the process of risky asset prices S.*

Proof. We start by writing

$$\mathbb{E}[(\Delta C^{\widehat{\xi}}_{t+1})^2 | \mathcal{F}_t] = \mathrm{Var}(\Delta C^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) + \mathbb{E}(\Delta C^{\widehat{\xi}}_{t+1} | \mathcal{F}_t)^2 \tag{2.10.11}$$

and try to find conditions for ξ and $V^{\widehat{\xi}}$ so that both terms on right-hand side minimize. By the definition of cost process, we have a useful expression

$$\Delta C^{\widehat{\xi}}_{t+1} = C^{\widehat{\xi}}_{t+1} - C^{\widehat{\xi}}_t = V^{\widehat{\xi}}_{t+1} - V^{\widehat{\xi}}_t - \widehat{\xi}^*_{t+1} \cdot \Delta S_{t+1}. \tag{2.10.12}$$

Now we can write the second term of the right-hand side of equation (2.10.11) as

$$\mathbb{E}(\Delta C^{\widehat{\xi}}_{t+1} | \mathcal{F}_t)^2 = [\mathbb{E}(V^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) - V^{\widehat{\xi}}_t - \widehat{\xi}_{t+1} \cdot \mathbb{E}(\Delta S_{t+1} | \mathcal{F}_t)]^2. \tag{2.10.13}$$

Assuming that t and $V^{\widehat{\xi}}_{t+1}$ are fixed, this is minimized when

$$V^{\widehat{\xi}}_t = \mathbb{E}(V^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) - \widehat{\xi}^*_{t+1} \cdot \mathbb{E}(\Delta S_{t+1} | \mathcal{F}_t). \tag{2.10.14}$$

If this is satisfied, we notice that

$$\mathbb{E}(\Delta C^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) = 0, \tag{2.10.15}$$

that is, $\widehat{\xi}$ is mean self-financing. Next let us consider the first term of the right-hand side of equation (2.10.11). Again we use expression (2.10.12) and then use basic properties of conditional variance and conditional covariance.

$$\begin{aligned}
\mathrm{Var}(\Delta C^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) &= \mathrm{Var}(V^{\widehat{\xi}}_{t+1} - V^{\widehat{\xi}}_t - \widehat{\xi}^*_{t+1} \cdot \Delta S_{t+1} | \mathcal{F}_t) \\
&= \mathrm{Var}(V^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) + \mathrm{Var}(\widehat{\xi}^*_{t+1} \cdot \Delta S_{t+1} | \mathcal{F}_t) \\
&- 2\mathrm{Cov}(V^{\widehat{\xi}}_{t+1},\ \widehat{\xi}^*_{t+1} \cdot \Delta S_{t+1} | \mathcal{F}_t) \\
&= \mathrm{Var}(V^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) + (\widehat{\xi}^*_{t+1})^{\mathsf{T}} \mathrm{Cov}(\Delta S_{t+1} | \mathcal{F}_t) \widehat{\xi}^*_{t+1} \\
&- 2\mathrm{Cov}(V^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t) \widehat{\xi}^*_{t+1}
\end{aligned} \tag{2.10.16}$$

We notice that right-hand side of (2.10.16) is of form $f(x) = c + x^{\mathsf{T}}Ax - 2b^{\mathsf{T}}x$ where the argument is vector $\widehat{\xi}^*_{t+1}$. Let us assume that such function is $f : \mathbb{R}^d \to \mathbb{R}$ and define derivative with respect to $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ as

$$\frac{\partial}{\partial x} f(x) = \left[\frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_d} f(x)\right]^{\mathsf{T}}.$$

Then we have $\frac{\partial}{\partial x} f(x) = 2Ax - 2b$, when A is symmetric matrix. Recall from linear algebra that f is minimized (we ignore explicit conditions) when $\frac{\partial}{\partial x} f(x) = 0$, that is $Ax - b = 0$. Hence we must have almost surely

$$\mathrm{Cov}(\Delta S_{t+1} | \mathcal{F}_t)\widehat{\xi}^*_{t+1} - \mathrm{Cov}(V^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t) = 0. \tag{2.10.17}$$

By equation (2.10.12) we can write

$$V^{\widehat{\xi}}_{t+1} = \Delta C^{\widehat{\xi}}_{t+1} + V^{\widehat{\xi}}_t + \widehat{\xi}^*_{t+1} \cdot \Delta S_{t+1} \tag{2.10.18}$$

Thus the second term on the right-hand side of equation (2.10.17) can be written as

$$\mathrm{Cov}(V^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t) = \mathrm{Cov}(\Delta C^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t) + \mathrm{Cov}(\Delta S_{t+1} | \mathcal{F}_t)\widehat{\xi}^*_{t+1}, \tag{2.10.19}$$

using the basic properties of conditional covariance. Hence the equation (2.10.17) is equivalently

$$\mathrm{Cov}(\Delta C^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t) = 0. \tag{2.10.20}$$

Recall that have the mean self-financing condition (equation (2.10.15)) for optimal $\widehat{\xi}$ so (2.10.20) yields

$$\mathbb{E}(\Delta C^{\widehat{\xi}}_{t+1} \Delta S_{t+1} | \mathcal{F}_t) = 0, \tag{2.10.21}$$

that is, the cost process of strategy $\widehat{\xi}$ is strongly orthogonal to S. □

The proof above gives a recursive way of solving the minimizing problem (2.10.3). Particularly equations (2.10.14) and (2.10.17) are useful. Let us restric ourselves to case, where there is only one risky asset in the market and put $V^{\widehat{\xi}}_T = X$ for derivative X. Then mentioned useful equations yield

$$V^{\widehat{\xi}}_t = \mathbb{E}(V^{\widehat{\xi}}_{t+1} | \mathcal{F}_t) - \widehat{\xi}^1_{t+1}\mathbb{E}(\Delta S_{t+1} | \mathcal{F}_t) \quad \text{with} \quad \widehat{\xi}^1_{t+1} = \frac{\mathrm{Cov}(V^{\widehat{\xi}}_{t+1},\ \Delta S_{t+1} | \mathcal{F}_t)}{\mathrm{Var}(\Delta S_{t+1} | \mathcal{F}_t)}, \tag{2.10.22}$$

that is, a recursive way to find locally risk-minimizing strategy. Here we put $\widehat{\xi}^0_t := V^{\widehat{\xi}}_t - \widehat{\xi}^1_t S^1_t$ to complete the strategy. However we have to assume further that there exist constant $c \in \mathbb{R}$ such that $(\mathbb{E}(\Delta S_t | \mathcal{F}_{t-1}))^2 \leq c\mathrm{Var}(\Delta S_t | \mathcal{F}_{t-1})$ almost surely in order to meet L^2-admissibility. See Proposition 10.10 in [10] for details.

Another minimizing problem arises, when we want to find self-financing strategy $\breve{\xi}$, that minimizes *quadratic hedging error*

$$\mathbb{E}[(X - V_T^{\breve{\xi}})^2] \tag{2.10.23}$$

for derivative X. Note that minimizing the quadratic hedging error corresponds to minimizing L^2-norm $\|X - V_T^{\breve{\xi}}\|_2$, which is the "distance" between X and $V_T^{\breve{\xi}}$. Recall that the value of self-financing strategy $\breve{\xi}$ can be written in terms of gain process as

$$V_t^{\breve{\xi}} = V_0^{\breve{\xi}} + G_t^{\breve{\xi}}. \tag{2.10.24}$$

Let us define set of strategies

$$\Xi := \{\xi \mid \xi \text{ is predictable, } G_t^{\xi} \in L^2(\mathbb{P}) \; \forall t \in \mathbb{T}\}.$$

Strategy $\breve{\xi} \in \Xi$ is called ***quadratic risk-minimizing***, if it minimizes quadratic hedging error for (European) derivative X, that is,

$$\mathbb{E}[(X - (V_0^{\breve{\xi}} + G_T^{\breve{\xi}}))^2] \leq \mathbb{E}[(X - (V_0^{\xi} + G_T^{\xi}))^2] \tag{2.10.25}$$

for all $\xi \in \Xi$. Solution to this minimizing problem is closely closely connected with local risk-minimizing problem. Let us again restrict to case with only one risky asset and one riskless asset in the market. Assume further that

$$\frac{\operatorname{Var}(\Delta S_t | \mathcal{F}_{t-1})}{\mathbb{E}(\Delta S_t | \mathcal{F}_{t-1})^2}$$

is deterministic for all $t \in \mathbb{T} \setminus \{0\}$. Then (we state without proving) the quadratic risk-minimizing strategy $\breve{\xi}$ is given by

$$V_0^{\breve{\xi}} := V_0^{\widehat{\xi}} \quad \text{and} \quad \breve{\xi}_t^1 := \widehat{\xi}_t^1 + \frac{\mathbb{E}(\Delta S_t | \mathcal{F}_{t-1})}{\mathbb{E}(\Delta S_t | \mathcal{F}_{t-1})^2 + \operatorname{Var}(\Delta S_t | \mathcal{F}_{t-1})} \big[V_{t-1}^{\widehat{\xi}} - (V_0^{\breve{\xi}} + G_{t-1}^{\breve{\xi}})\big], \tag{2.10.26}$$

where $\widehat{\xi}$ refers to locally risk-minimizing strategy given in (2.10.22).

Another imperfect way to hedge derivatives in incomplete market would be ***super-hedging*** which we already encountered with in Section 2.8. In super-hedging, we try to find the cheapest (by initial price) self-financing strategy that surpasses the payoff of the derivative. The upside of this approach is, that we are always (at least in theory) able to stay on the "safe side", that is, there is no risk of losing money more than was planned beforehand. The downside of this approach is that even the cheapest super-hedge may still require a large initial investment so it is expensive compared to other methods. More about super-hedging can be read from Chapter 7 of [10].

Assuming that investor does not want to pay the initial amount of capital required to build a superhedge and he/she is willing to carry some risk, there is some alternatives to consider. We will introduce two of these alternatives. In ***quantile hedging*** one tries to find self-financing strategy ξ that maximizes the probability of successful super-hedge $\mathbb{P}(V_T^\xi \geq X)$ over the strategies that cost (initially) less than super-hedging strategy. Yet another imperfect hedging strategy is called ***shortfall risk-minimizing***. In this method, one tries to find strategy ξ that minimizes $\mathbb{E}[(X - V_T^\xi)^+]$. We will not determine how to build these alternative strategies. More about these approaches can be found from Chapter 8 of [10].

2.11 Change of numeraire

The existence of a concept introduced in the next important lemma is well-known result called *Radon-Nikodym theorem*. We will omit the proof.

Lemma 2.11.1. *Let $\mathbb{Q} \sim \mathbb{P}$ be two probability measures on $(\Omega, \mathcal{F})$. Then there exists random variable $Z \geq 0$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(ZX) \quad \forall X \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$$

We denote

$$Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$$

and call Z the Radon-Nikodym derivative of measure $\mathbb{Q}$ with respect to measure $\mathbb{P}$.

Remark 2.11.2. (i) Expectation under $\mathbb{P}$ of Radon-Nikodym derivative Z is one, since

$$\mathbb{E}_{\mathbb{P}}(Z) = \int_\Omega Z(\omega) d\mathbb{P}(\omega) = \int_\Omega \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) = \mathbb{Q}(\Omega) = 1.$$

(ii) In the finite (or countable) sample space:

$$Z(\omega) = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \frac{\mathbb{Q}(\{\omega\})}{\mathbb{P}(\{\omega\})} \quad \forall \omega \in \Omega.$$

△

Lemma 2.11.3. *(Bayes' formula) Let $\mathbb{Q} \sim \mathbb{P}$ be two probability measures on $(\Omega, \mathcal{F})$ and assume that Z is Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, then*

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(XZ|\mathcal{G})}{\mathbb{E}_{\mathbb{P}}(Z|\mathcal{G})}, \quad \forall X \in L^1(\Omega, \mathcal{F}, \mathbb{Q}). \tag{2.11.4}$$

Proof. We have prove to that $\mathbb{E}_{\mathbb{P}}(XZ|\mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G})\mathbb{E}_{\mathbb{P}}(Z|\mathcal{G})$. We prove the aforementioned by the definition of conditional expectation 2.1.10. Let $G \in \mathcal{G}$, by the properties of conditional expectation and Radon-Nikodym derivative, we get

$$\begin{aligned}\int_G \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G})\mathbb{E}_{\mathbb{P}}(Z|\mathcal{G}) d\mathbb{P} &= \int_\Omega \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) Z \mathbb{1}_G|\mathcal{G}] d\mathbb{P} = \int_\Omega \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) Z \mathbb{1}_G d\mathbb{P} \\ &= \int_\Omega \mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_G|\mathcal{G}) d\mathbb{Q} = \int_\Omega X \mathbb{1}_G d\mathbb{Q} = \int_G XZ d\mathbb{P}.\end{aligned}$$

□

In this section let us write $Y = \{Y_t \mid t \in \mathbb{T}\}$ for the value of an arbitrary self-financing and predictable strategy or one of the assets $S^1, \dots, S^d$. From the Proposition 2.4.16 we know that if $\mathbb{Q}$ is an equivalent martingale measure with numeraire S^0, then $\widetilde{Y} = \{Y_t/S_t^0 \mid t \in \mathbb{T}\}$ is $\mathbb{Q}$-martingale. The next proposition will show how a new measure $\mathbb{Q}^Y$ with numeraire Y should be defined so that it becomes equivalent martingale measure in the market model.

Proposition 2.11.5. *In an arbitrage-free market model, let $\mathbb{Q}$ be an equivalent martingale measure with numeraire S^0. Let also $Y = \{Y_t \mid t \in \mathbb{T}\}$ be a positive process such that $\widetilde{Y}$ is a $\mathbb{Q}$-martingale. Then the measure $\mathbb{Q}^Y$ defined by*

$$\frac{d\mathbb{Q}^Y}{d\mathbb{Q}} = \left(\frac{S_0^0}{Y_0}\right)\frac{Y_T}{S_T^0} \tag{2.11.6}$$

is such that

$$Y_t \mathbb{E}_{\mathbb{Q}^Y}\left(\frac{X}{Y_T}|\mathcal{F}_t\right) = S_t^0 \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{S_T^0}|\mathcal{F}_t\right), \quad t \in \mathbb{T} \tag{2.11.7}$$

for every $X \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$. Therefore $\mathbb{Q}^Y$ is an equivalent martingale measure with numeraire Y.

Proof. Let us denote

$$Z_t = \left(\frac{S_0^0}{Y_0}\right)\frac{Y_t}{S_t^0}, \text{ for } t \in \mathbb{T}.$$

Since Y discounted with S^0 is $\mathbb{Q}$-martingale, we have $\mathbb{E}_{\mathbb{Q}}(Z_T|\mathcal{F}_t) = Z_t$. Now by Bayes' formula (Lemma 2.11.3) we get

$$Y_t \mathbb{E}_{\mathbb{Q}^Y}\left(\frac{X}{Y_T}|\mathcal{F}_t\right) = Y_t \frac{\mathbb{E}_{\mathbb{Q}}(\frac{X}{Y_T} Z_T|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(Z_T|\mathcal{F}_t)} = \frac{Y_t}{Z_t}\frac{S_0^0}{Y_0}\mathbb{E}_{\mathbb{Q}}\left(\frac{X}{S_T^0}|\mathcal{F}_t\right) = S_t^0 \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{S_T^0}|\mathcal{F}_t\right)$$

Indeed we have

$$\mathbb{E}_{\mathbb{Q}^Y}\left(\frac{S_T}{Y_T}|\mathcal{F}_t\right) = \frac{S_t^0}{Y_t}\mathbb{E}_{\mathbb{Q}}\left(\frac{S_T}{S_T^0}|\mathcal{F}_t\right) = \frac{S_t^0}{Y_t}\cdot\frac{S_t}{S_t^0} = \frac{S_t}{Y_t},$$

so $\mathbb{Q}^Y$ is equivalent martingale measure with Y as numeraire.

□

In earlier section, we have restricted ourselves to deterministic interest rates. In reality this is not the case but interest rates are stochastic. That makes pricing the derivatives more challenging, since we have to consider the joint distribution discounting factor and the payoff of the derivative, when we are computing the expectation under equivalent martingale measure. An important concept called *forward measure* can sometimes ease the computation. The forward measure uses suitable *bond* as a numeraire so that the need for joint distribution disappears.

As we already mentioned, numeraire Y does not have to be a single asset. It can be entire portfolio of asset. This is particularly useful, when one tries to price a *swap* *option* (or *swaption*), that is, an option giving its owner the right but not the obligation to exchange cash flows by contract with another party. In this situation a convenient choice of numeraire could be a portfolio consisting of *zero-coupon bonds* maturing at times different times. We will not review these concepts in discrete-time market models.

Example 2.11.8. Consider market model defined in Example 2.2.4, where we have two asset S and B. Recall that in another Example 2.2.4, we found equivalent martingale measure $\mathbb{Q}$ with numeraire B. Measure $\mathbb{Q}$ is unique and symmetric with

$$\mathbb{Q}(\{\omega\}) = \frac{1}{4}, \text{ for each } \omega \in \Omega = \{(0,0),(1,0),(0,1),(1,1)\}.$$

One might ask: does there exist equivalent martingale measure with S as numeraire? The answer is yes, since $S_t(\omega) > 0$ for all $t = 0, 1, 2$ and $\omega \in \Omega$, furthermore, $\widetilde{S} = \frac{S}{B}$ is a $\mathbb{Q}$-martingale. Let us denote such measure by $\mathbb{Q}^S$, now by Proposition 2.11.5 we have the relationship:

$$\frac{\mathbb{Q}^S(\{\omega\})}{\mathbb{Q}(\{\omega\})} = \left(\frac{B_0}{S_0}\right)\frac{S_2(\omega)}{B_2} \quad \Longleftrightarrow \quad \mathbb{Q}^S(\{\omega\}) = \frac{1}{726}S_2(\omega),$$

which yields values

$$\mathbb{Q}^S(\{\omega\}) = \begin{cases} \frac{16}{121}, & \omega = (0,0) \\ \frac{28}{121}, & \omega \in \{(1,0),(0,1)\} \\ \frac{49}{121}, & \omega = (1,1). \end{cases}$$

We notice that received $\mathbb{Q}^S$-probabilities indeed are positive and sum up to one which is essential. △

2.12 American options

The American options generalize the European options in such way that they can be exercised at any moment before (and including) the expiration date T. This feature makes it trickier to price and hedge American-type options. The buyer of the option wants to find the best (regarding the payoff) moment to exercise the option. The seller, for his part, tries to hedge against the buyer's payoff. We will study both (buyer's and seller's) standpoints. Let us first define a general American derivative.

Definition 2.12.1. An American derivative is a non-negative stochastic process $X = \{X_t \,|\, t \in \mathbb{T}\}$ adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t^S \,|\, t \in \mathbb{T}\}$. ◊

We are particularly interest in the following types of derivatives. An *American call* and *put option* are like the European counterparts but they give right (but again no obligation) to exercise the option at any time $t \in \mathbb{T}$. We denote the payoff of a call option C at time $t \in \mathbb{T}$ by C_t and a general risky asset by S^i hence

$$C_t = (S^i_t - K)^+,$$

where K is the strike price introduced in Section 2.7. We require that the decision of whether to exercise or not is based on the information available at time t. Hence we make the next natural definition.

Definition 2.12.2. An *exercise strategy* (or *time*) is a stopping time, namely a random variable $\tau : \Omega \to \mathbb{T}$ such that $\{\tau = t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$. We denote by Υ the family of all exercise strategies. $\Diamond$

Contrary to fixed exercise time T of the European options, now we have random exercise time depending on the changes of asset prices. For every scenario $\omega \in \Omega$, the number $\tau(\omega)$ represents the moment when investor decides to exercise the American option.

Definition 2.12.3. Let $Y = \{Y_t \mid t \in \mathbb{T}\}$ be a stochastic process and τ a stopping time on the same stochastic basis. Then we denote by Y^τ the process Y stopped at τ defined by $Y^\tau_t := Y_{\tau\wedge t} := Y_{\min(\tau,t)}$ for every $t \in \mathbb{T}$. Explicitly we define $Y_{\tau\wedge 0} = Y_0$ and for $t \in \mathbb{T} \setminus \{0\}$:

$$Y_{\tau\wedge t} := \sum_{u=0}^{t-1} Y_u \mathbb{1}_{\{\tau=u\}} + Y_t \mathbb{1}_{\{\tau\geq t\}} \tag{2.12.4}$$

Process Y^τ is often called shortly a *stopped process.* $\Diamond$

Let us again assume that $Y = \{Y_t \mid t \in \mathbb{T}\}$ is a stochastic process and τ is a stopping time on the same stochastic basis. Notice that mapping $(t, \omega) \mapsto Y^{\tau(\omega)}_t = Y_{\tau(\omega)\wedge t}$ defines a process, while mapping $\omega \mapsto Y_{\tau(\omega)}(\omega)$ defines a random variable with values $Y_{\tau(\omega)}(\omega) = \sum_{t=0}^{T} Y_t(\omega)\mathbb{1}_{\{\tau(\omega)=t\}}$. In the next lemma we will use notation $\mathcal{F}_\tau$ for a *stopped sigma-algebra* defined by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau = t\} \in \mathcal{F}_t \text{ for every } t \in \mathbb{T}\}. \tag{2.12.5}$$

The stopped sigma-algebra $\mathcal{F}_\tau$ is indeed a sigma-algebra. We omit the proof but mention, though, that it is fairly straightforward. $\mathcal{F}_\tau$ can be interpret as the collection of all events which are known to us at time τ.

Lemma 2.12.6. *(Optional sampling and stopping for bounded stopping times) Let us assume that M is a martingale and N is a supermartingale. Further, suppose that τ and λ are bounded stopping times with $\lambda \leq \tau$ almost surely. Then we have*

(i) optional sampling: $\mathbb{E}(M_\tau|\mathcal{F}_\lambda) = M_\lambda$ and $\mathbb{E}(N_\tau|\mathcal{F}_\lambda) \leq N_\lambda$,

(ii) optional stopping: M^τ is a martingale and N^τ is a supermartingale.

Proof. We will only prove part (ii) to point out an interesting connection between a martingale transformation and a stopped process. For part (i) see, for example, the proof of Theorem 5.2.7 in [9]. We begin the proof of part (ii) by writing $\mathbb{1}_{\{\tau=u\}} = \mathbb{1}_{\{\tau\geq u\}} - \mathbb{1}_{\{\tau\geq u+1\}}$ for the stopped process M^τ:

$$\begin{aligned} M_{\tau\wedge t} &= \sum_{u=0}^{t-1} M_u \mathbb{1}_{\{\tau=u\}} + M_t \mathbb{1}_{\{\tau\geq t\}} = \sum_{u=0}^{t-1} M_u \mathbb{1}_{\{\tau\geq u\}} - \sum_{u=0}^{t-1} M_u \mathbb{1}_{\{\tau\geq u+1\}} + M_t \mathbb{1}_{\{\tau\geq t\}} \\ &= \sum_{u=0}^{t} M_u \mathbb{1}_{\{\tau\geq u\}} - \sum_{u=1}^{t} M_{u-1} \mathbb{1}_{\{\tau\geq u\}} = M_0 + \sum_{u=1}^{t} \mathbb{1}_{\{\tau\geq u\}} \Delta M_u. \end{aligned} \tag{2.12.7}$$

We observe that if M is a martingale, then the stopped process M^τ is just a special case of martingale transformation $M^\tau_t = M_0 + (\phi \bullet M)_t$ with a predictable bounded non-negative process defined by $\phi_u = \mathbb{1}_{\{\tau\geq u\}}$. Now Proposition 2.4.8 yields the result. This means that we can not beat the system even with a smart exercise strategy. □

Recall that we defined stopping times in such way that they are always bounded. Previous lemma can be, however, generalized for unbounded stopping times by requiring that M is *uniformly integrable* martingale. The generalized version is beyond the scope of our aspect. More about this topic can be found, for example, from Chapter A14 in [19] and Chapter 5.3 of [9].

Definition 2.12.8. If X is an American derivative and $\tau \in \Upsilon$ is an exercise strategy, then the random variable X_τ is called the payoff of X relative to the strategy τ. Furthermore, we say that an exercise strategy ν is *optimal* for X with respect to $\mathbb{Q}$ if

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{X}_\nu) = \max_{\tau\in\Upsilon} \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_\tau).$$

◊

In the case of arbitrage-free and complete market, we could replicate the European derivative and determine the price via replicating strategy. Unfortunately it is not possible to construct a replicating strategy for the American derivative X in general even if market is complete. We can though determine lower and upper bound for the price of X in a similar fashion as in Lemma 2.8.7. Let us again denote by

$$\Theta^+_X = \{\theta \in \Theta \,|\, V^\theta_t \geq X_t, \forall t \in \mathbb{T}\} \quad \text{and} \quad \Theta^-_X = \{\theta \in \Theta \,|\, \exists \tau \in \Upsilon : V^\theta_\tau \leq X_\tau\}$$

the family of super-replicating strategies and the family of strategies that can be exploited by using an exercise strategy which guarantees greater (or equal) payoff for X_τ, respectively. The advantage of the latter strategies can be achieved by taking a short position of such portfolio to receive funds for purchasing the American option.

Let us denote by π^X the initial price of X. Due to earlier discussion, it must hold that $\pi^X \geq V^\theta_0$ for any $\theta \in \Theta^-_X$ to prevent the existence of an arbitrage. Evidently we must also have $\pi^X \leq V^\theta_0$ for any $\theta \in \Theta^+_X$. The next result concludes that the arbitrage-free value of the payoff relative to optimal exercise strategy is included in the following interval.

Proposition 2.12.9. *In an arbitrage-free market*

$$\sup_{\theta\in\Theta^-_X} V^\theta_0 \leq \max_{\tau\in\Upsilon} \mathbb{E}_{\mathbb{Q}}(\widetilde{X}_\tau) \leq \inf_{\theta\in\Theta^+_X} V^\theta_0,$$

for every equivalent martingale measure $\mathbb{Q}$.

Proof. Let us assume that $\theta \in \Theta_X^-$, then there exists an exercise strategy $\lambda \in \Upsilon$ such that $V_\lambda^\theta \leq X_\lambda$. We define $\kappa(\omega) = 0$, $\forall \omega \in \Omega$, which is clearly a bounded stopping time (since $\{\kappa(\omega) = 0\} = \Omega \in \mathcal{F}_0$ and $\{\kappa(\omega) = t\} = \emptyset \in \mathcal{F}_t$ for every $t \in \mathbb{T} \setminus \{0\}$). Recall that $\widetilde{V}^\theta$ is a $\mathbb{Q}$-martingale thus by optional sampling (Lemma 2.12.6 part (i)), we have

$$V_0^\theta = \widetilde{V}_0^\theta = \mathbb{E}_\mathbb{Q}(\widetilde{V}_\lambda^\theta | \mathcal{F}_0) = \mathbb{E}_\mathbb{Q}(\widetilde{V}_\lambda^\theta) \leq \mathbb{E}_\mathbb{Q}(\widetilde{X}_\lambda^\theta) \leq \max_{\tau \in \Upsilon} \mathbb{E}_\mathbb{Q}(\widetilde{X}_\tau)$$

Similar reasoning yields another inequality for the case $\theta \in \Theta_X^+$. □

Next we will introduce some very important concepts that help us in our task of finding optimal exercise strategy for buyer and hedge for seller.

Definition 2.12.10. Given an adapted process $X = \{X_t \mid t \in \mathbb{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we say that the smallest $(\mathbb{F}, \mathbb{P})$-supermartingale that dominates X is the *Snell envelope* of X with respect to $\mathbb{P}$.

Explicitly, if Z is the Snell envelope of X, then $Z_t \geq X_t$ for each $t \in \mathbb{T}$ and $\widehat{Z}_t \geq Z_t$ for each $t \in \mathbb{T}$ for any other $(\mathbb{F}, \mathbb{P})$-supermartingale $\widehat{Z}$ that has $\widehat{Z}_t \geq X_t$ for all $t \in \mathbb{T}$. ◇

Lemma 2.12.11. *Given an American derivative X, the process $\widetilde{Z}$ recursively defined by*

$$\widetilde{Z}_t = \begin{cases} \widetilde{X}_T, & t = T \\ \max\{\widetilde{X}_t, \mathbb{E}_\mathbb{Q}(\widetilde{Z}_{t+1} | \mathcal{F}_t)\}, & t \in \mathbb{T} \setminus \{T\} \end{cases} \tag{2.12.12}$$

is the Snell envelope of $\widetilde{X}$ with respect to $\mathbb{Q}$.

Proof. Clearly the way $\widetilde{Z}_t$ is defined as the maximum of X_t and some other term, forces $\widetilde{Z}_t \geq \widetilde{X}_t$ for every $t \in \mathbb{T}$, that is, $\widetilde{Z}$ dominates $\widetilde{X}$. Also by the same maximum, we have $\widetilde{Z}_t \geq \mathbb{E}_\mathbb{Q}(\widetilde{Z}_{t+1} | \mathcal{F}_t)$ so Z is a supermartingale under $\mathbb{Q}$. Let us then show, by backward induction, that Z is the smallest of dominating supermartingales. Suppose that $\widetilde{Y}$ is a $\mathbb{Q}$-supermartingale that also dominates $\widetilde{X}$. At time T we have (the base case) $\widetilde{Z}_T = \widetilde{X}_T \leq \widetilde{Y}_T$, since $\widetilde{Y}$ dominates $\widetilde{X}$. We make induction hypothesis that $\widetilde{Z}_{t+1} \leq \widetilde{Y}_{t+1}$, hence

$$\widetilde{Z}_t = \max\{\widetilde{X}_t, \mathbb{E}_\mathbb{Q}(\widetilde{Z}_{t+1} | \mathcal{F}_t)\} \leq \max\{\widetilde{X}_t, \mathbb{E}_\mathbb{Q}(\widetilde{Y}_{t+1} | \mathcal{F}_t)\} \leq \max\{\widetilde{X}_t, \widetilde{Y}_t\} = \widetilde{Y}_t, \tag{2.12.13}$$

where the second inequality comes from the fact that $\widetilde{Y}$ is a $\mathbb{Q}$-supermartingale. □

Lemma 2.12.14. *(Doob's decomposition) Let us assume that $Y = \{Y_t \mid t \in \mathbb{T}\}$ is an $\mathbb{F}$-adapted stochastic process with $Y_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in \mathbb{T}$. Then there exist a $(\mathbb{F}, \mathbb{P})$-martingale $M = \{M_t \mid t \in \mathbb{T}\}$ and a predictable process $A = \{A_t \mid t \in \mathbb{T}\}$ with $A_0 = 0$ so that*

$$Y_t = M_t + A_t \quad \textit{for all } t \in \mathbb{T}. \tag{2.12.15}$$

This is called the Doob's decomposition of Y (with respect to $\mathbb{P}$).

Proof. Let us, at first, assume that such A and M exists and try to solve these with respect to Y. From equation 2.12.15 we get $\Delta Y_t = \Delta M_t + \Delta A_t$ and thus

$$\mathbb{E}(\Delta Y_t|\mathcal{F}_{t-1}) = \mathbb{E}(\Delta M_t|\mathcal{F}_{t-1}) + \mathbb{E}(\Delta A_t|\mathcal{F}_{t-1}) = \Delta A_t, \tag{2.12.16}$$

since M is martingale and A is predictable. Now we can solve A with respect to Y recursively:

$$\begin{aligned} A_t &= A_{t-1} + \mathbb{E}(\Delta Y_t|\mathcal{F}_{t-1}) \\ &= A_{t-2} + \mathbb{E}(\Delta Y_{t-1}|\mathcal{F}_{t-2}) + \mathbb{E}(\Delta Y_t|\mathcal{F}_{t-1}) \\ &= \cdots \\ &= A_0 + \sum_{u=1}^{t} \mathbb{E}(\Delta Y_u|\mathcal{F}_{u-1}) = \sum_{u=1}^{t} \mathbb{E}(\Delta Y_u|\mathcal{F}_{u-1}). \end{aligned} \tag{2.12.17}$$

We observe that A is indeed predictable. Now M must satisfy

$$M_t = Y_t - A_t = Y_t - \sum_{u=1}^{t} \mathbb{E}(\Delta Y_u|\mathcal{F}_{u-1}), \tag{2.12.18}$$

consequently M is a martingale, since

$$\mathbb{E}(\Delta M_t|\mathcal{F}_{t-1}) = \mathbb{E}(\Delta Y_t|\mathcal{F}_{u-1}) - \Delta A_t = 0. \tag{2.12.19}$$

Let us now prove uniqueness of Doob's decomposition. Suppose that Y has two decompositions $Y_t = M_t + A_t$ and $Y_t = \hat{M}_t + \hat{A}_t$. Hence we have $A_t - \hat{A}_t = \hat{M}_t - M_t$. Recall that $\hat{M} - M$ is a martingale (by Lemma 2.4.5), so $A - \hat{A}$ is a predictable martingale. This means that for every $t \in \mathbb{T} \setminus \{0\}$ the following holds

$$A_t - \hat{A}_t = \mathbb{E}(A_t - \hat{A}_t|\mathcal{F}_{t-1}) = A_{t-1} - \hat{A}_{t-1}. \tag{2.12.20}$$

By starting from $A_0 = \hat{A}_0 = 0$ and using the recursion (2.12.20) we get $A_1 = \hat{A}_1$ and thus $A_2 = \hat{A}_2$. We can continue the procedure so that $A_t = \hat{A}_t$ for every $t \in \mathbb{T}$. Uniqueness of A ensures uniqueness of M. □

Let us consider Doob's decomposition $Y = M + A$. We notice that process A determines whether the process Y is supermartingale, martingale or submartingale. If mapping $t \mapsto A_t$ is decreasing ($\mathbb{P}$-almost surely), then Y is a supermartingale. If the mapping is increasing, then Y is a submartingale. Also Y is a martingale if and only if $A_t = 0$ for every $t \in \mathbb{T}$.

Consider now the Snell envelope Z for an American derivative X, that is, smallest supermartingale that dominates X. Suppose that $Z_t = M_t + A_t$ is the Doob's decomposition of Z. Since $t \mapsto A_t$ is decreasing, we have $M_t \geq Z_t \geq X_t$ for every $t \in \mathbb{T}$. If we assume that a market is arbitrage-free and complete, then martingale M can be represented, by Proposition 2.9.5, in the form

$$M_t = Z_0 + \sum_{u=1}^{t} \phi_u^* \cdot \Delta \widetilde{S}_u, \tag{2.12.21}$$

where ϕ^* is some d-dimensional predictable process. Here d corresponds to the number of risky assets. We can choose deposits ϕ^0 to bank account in a such way that strategy $\phi := (\phi^0, \phi^*)$ becomes self-financing with the initial investment Z_0. Strategy ϕ is an (American) super-hedge for X meaning that the value of this strategy satisfies $V_t^\phi \geq X_t$ for all $t \in \mathbb{T}$. Using the strategy ϕ and withdrawing funds according to A results a cash flow Z. Actually, Z_t is the minimal amount of money required to super-hedge X at time t. See, for example, Chapter 6.1 of [10] for details.

Let us now consider the optimal moment for buyer to exercise an American option. Suppose that Z is the Snell envelope of an American option X with $X_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in \mathbb{T}$. We define an exercise strategy by

$$\nu_{\min} := \min\{t \in \mathbb{T} \mid Z_t = X_t\}. \tag{2.12.22}$$

So $\nu_{\min}$ corresponds to first time the *face-value* (the value paid for the buyer at exercise time) of an option reaches its Snell envelope. Again we can decomposite Z in to $Z_t = M_t + A_t$ by Doob's decomposition. We define another exercise strategy by

$$\nu_{\max} := \inf\{t \in \mathbb{T} \mid A_{t+1} \neq 0\}. \tag{2.12.23}$$

Recall from our earlier discussion that A determines whether the underlying process is a martingale. So $\nu_{\max}$ corresponds to the first time, that the Snell envelope of X loses its martingale property. These two exercise strategies ($\nu_{\min}$ and $\nu_{\max}$) can be proven to be optimal as in Definition 2.12.8. Particularly "min" and "max" refer to the fact that these are the first and the last optimal exercise strategies, respectively. The next theorem states the aforesaid claim.

Theorem 2.12.24. *Random variables $\nu_{\min}$ and $\nu_{\max}$ defined by (2.12.22) and (2.12.23) are optimal exercise strategies. Furthermore, if $\nu \in \Upsilon$ is any other optimal exercise strategy, then*

$$\nu_{\min} \leq \nu \leq \nu_{\max}.$$

Proof. See, for example, Chapter 6.2 of [10] or Chapter 5.4 of [9]. □

2.13 Binomial model

In previous sections we constructed martingale methods for pricing of derivatives in discrete-time market. In this section we are going to use these methods for pricing in the Cox-Ross-Rubinstein binomial market model. Actually, we have already had the foretaste of (two-period) binomial market model in Example 2.2.4 and in later examples with the same underlying market model. The Cox-Ross-Rubinstein binomial model was introduced in 1979 (see [5]). We will not follow the original paper, though.

2.13.1 Market model

Let us again assume that $\mathbb{T} = \{0, 1, 2, \ldots, T\}$ for some $T \in \mathbb{N}$. We also assume that $d = 1$ so that, there is only one stock $S = S^1$ and a riskless asset $B = S^0$ in the market. We

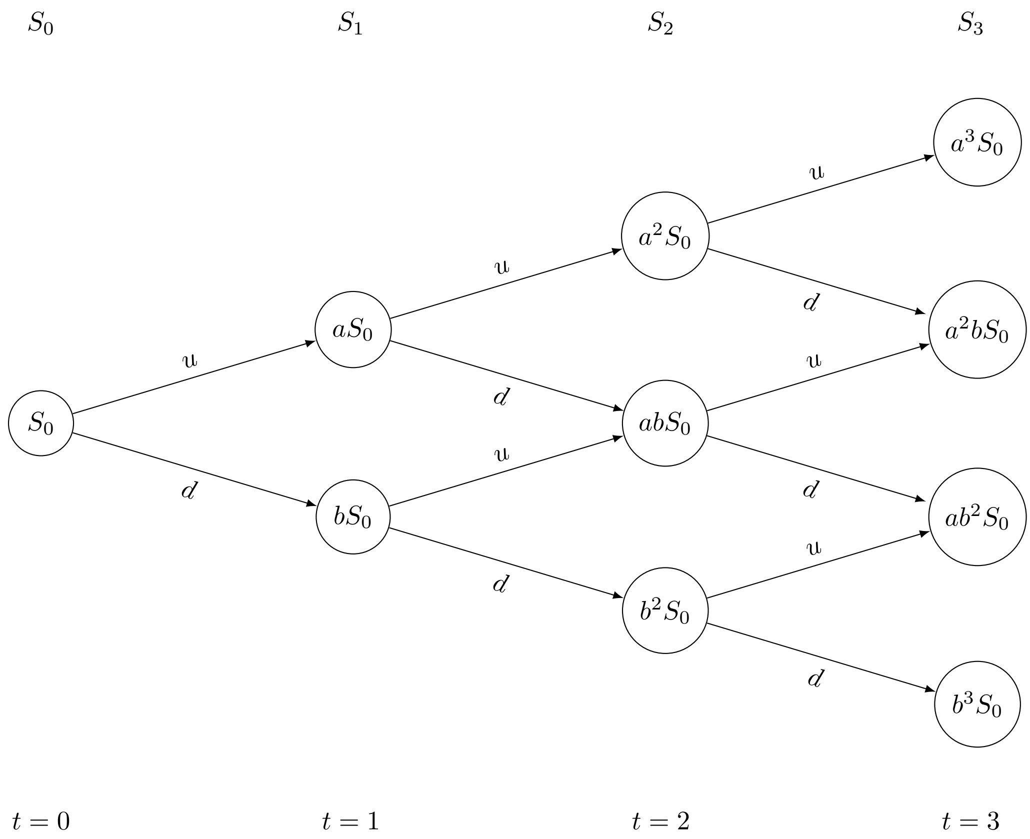

Figure 2.3: Price dynamics of a single stock S in a binomial model with three periods. For the sake of simplicity, we have left out the term $+1$ from the multipliers of S_0. The diagram should be interpret, for example, in a following way: in S_3-column a^2bS_0 stands for $(1+u)^2(1+d)S_0$. Generally $a = 1+u$ (up) and $b = 1+d$ (down).

choose B as a numeraire. The dynamics of the assets are given by

$$B_0 = 1 \quad \text{and} \quad B_t = (1+r)^t,\ t \in \mathbb{T} \setminus \{0\}, \tag{2.13.1}$$

for some fixed $r > 0$, and

$$S_0 > 0 \text{ (constant)} \quad \text{and} \quad S_t = (1+R_t)S_{t-1}, \quad t \in \mathbb{T} \setminus \{0\} \tag{2.13.2}$$

where random variable R_t takes only the values u (up) and d (down), so that

$$R_t = \begin{cases} u, & \text{with probability } p \in (0,1) \\ d, & \text{with probability } 1-p \end{cases}$$

where $-1 < d < u$ are fixed. The dynamics of a risky asset in three periods are visualized in Figure 2.3. We choose conveniently the sample space

$$\Omega = \{(\omega_1, \omega_2, \ldots, \omega_T) \in \mathbb{R}^T \mid \omega_t \in \{d, u\},\ t \in \mathbb{T} \setminus \{0\}\}$$

and the natural filtration $\mathbb{F}$ generated by the stock price values, that is $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t = \sigma(S_k \mid k \leq t)$ for $t \in \mathbb{T} \setminus \{0\}$. Furthermore, $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ is the power set of Ω. We also define

$$\mathbb{P}(\{\omega\}) = \mathbb{P}(R_t = \omega_t,\, t \in \mathbb{T} \setminus \{0\}),$$

for $\omega = (\omega_1, \omega_2, \ldots, \omega_T) \in \Omega$.

Lemma 2.13.3. *In binomial model the condition $d < r < u$ must hold so that equivalent martingale measure can exist.*

Proof. Let us assume that a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is an equivalent martingale measure in the binomial model. Thus by definition

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{S}_t | \mathcal{F}_{t-1}) = \widetilde{S}_{t-1}$$

which can be equivalently written as

$$\mathbb{E}_{\mathbb{Q}}(S_t | \mathcal{F}_{t-1}) = (1+r) S_{t-1},$$

and since S_{t-1} is $\mathcal{F}_{t-1}$-measurable

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{S_t - S_{t-1}}{S_{t-1}} | \mathcal{F}_{t-1}\right) = r$$

hence taking the expectation on both sides yields

$$\mathbb{E}_{\mathbb{Q}}(R_t) = r.$$

Since R_t is defined to have solely two realizations d and u, its expectation can equal r only if $d < r < u$. $\square$

Proposition 2.13.4. *Discounted stock price process $\widetilde{S}$ is a $\mathbb{Q}$-martingale if and only if random variables $R_1, R_2, \ldots, R_T$ (rates of return) are independent and identically distributed (i.i.d.), with $\mathbb{Q}(R_1 = u) = q$ and $\mathbb{Q}(R_1 = d) = 1 - q$, where $q = (r-d)/(u-d)$.*

Proof. "$\Leftarrow$": If $R_1, R_2, \ldots, R_T$ are i.i.d. (with the aforementioned distribution), then

$$\mathbb{E}_{\mathbb{Q}}(R_t | \mathcal{F}_{t-1}) = \mathbb{E}_{\mathbb{Q}}(R_t) = uq + d(1-q) = q(u-d) + d = r$$

This means that $\widetilde{S}$ is a $\mathbb{Q}$-martingale (see the proof of Lemma 2.13.3 backwards).

"$\Rightarrow$": If $\widetilde{S}$ is a $\mathbb{Q}$-martingale, then (see again the proof of Lemma 2.13.3) we have $\mathbb{E}_{\mathbb{Q}}(R_t | \mathcal{F}_{t-1}) = r$, thus

$$d\, \mathbb{Q}(R_t = d | \mathcal{F}_{t-1}) + u\, \mathbb{Q}(R_t = u | \mathcal{F}_{t-1}) = r,$$

since R_t have realizations d and u. We also have

$$\mathbb{Q}(R_t = d | \mathcal{F}_{t-1}) + \mathbb{Q}(R_t = u | \mathcal{F}_{t-1}) = 1,$$

so by writing $q = \mathbb{Q}(R_t = u|\mathcal{F}_{t-1})$ we get

$$d(1-q) + uq = r \qquad \Longleftrightarrow \qquad q = \frac{r-d}{u-d}.$$

We notice that q does not depend on time variable t. The independency of $R_1, R_2, \dots, R_T$ can be confirmed inductively, since for $\omega = (\omega_1, \omega_2, \dots, \omega_T)$, $\omega \in \Omega$ we have

$$\begin{aligned}\mathbb{Q}(R_1 = \omega_1, \dots, R_T = \omega_T) &= \mathbb{Q}(R_1 = \omega_1, \dots, R_{T-1} = \omega_{T-1})\mathbb{Q}(R_T = \omega_T|\mathcal{F}_{T-1}) \\ &= \dots = \prod_{j=1}^{T} q_j, \quad q_j = \begin{cases} q, & \omega_j = u \\ 1-q, & \omega_j = d \end{cases}\end{aligned}$$

Hence $R_1, R_2, \dots, R_T$ are i.i.d. as we stated. □

Recall that the first fundamental theorem of asset pricing (Theorem 2.4.15) states that the market model is arbitrage-free if there exist at least one equivalent martingale measure. Furthermore, the second fundamental theorem (Theorem 2.9.3) states that if the equivalent martingale measure is unique, then the market model is complete. We have observed that under the conditions given in Lemma 2.13.3 and Proposition 2.13.4, the Cox-Ross-Rubinstein **binomial market model is both arbitrage-free and complete**, since the unique equivalent martingale measure exists.

2.13.2 Option valuation

Since the (properly defined) binomial market model is arbitrage-free and complete, we can derive formulas for the prices of options by calculating (conditional) expectations of discounted payoffs under the equivalent martingale measure (as in Theorem 2.8.9). For calculations, we need to know the $\mathbb{Q}$-distibution of stock price at time t. From (2.13.2) we can deduce that

$$S_t = S_0 \prod_{j=1}^{t} (1+R_j) \quad \text{and} \quad S_T = S_t \prod_{j=t+1}^{T} (1+R_j)$$

for $t \in \mathbb{T} \setminus \{0, T\}$. The price of the stock at time t depends only of the number of up-moves between time 0 and t. If we assume that $R_1, R_2, \dots, R_t$ are i.i.d, then under the measure $\mathbb{Q}$ defined in Proposition 2.13.4 S_t is a random variable which follows the binomial distribution:

$$\mathbb{Q}(S_t = (1+u)^k(1+d)^{t-k}S_0) = \binom{t}{k} q^k (1-q)^{t-k} \tag{2.13.5}$$

for $k = 0, 1, \dots, t$ and furthermore

$$\mathbb{Q}(S_T = (1+u)^k(1+d)^{T-t-k}S_t \mid \mathcal{F}_t) = \binom{T-t}{k} q^k (1-q)^{T-t-k} \tag{2.13.6}$$

for $k = 0, 1, \ldots, T-t$ which have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us assume that derivative is path independent and of form $X = f(S_T)$ for some function f and that $\mathbb{Q}$ is satisfies the conditions of Proposition 2.13.4. Then the time t price of such derivative is

$$\begin{aligned} c_t^X &= B_t \mathbb{E}_{\mathbb{Q}}(f(S_T)/B_T|\mathcal{F}_t) \\ &= \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} f((1+u)^k (1+d)^{T-t-k} S_t), \end{aligned} \tag{2.13.7}$$

where 2.13.6 gave us the conditional distribution of S_T. Let us now consider European call option with strike price K. Recall that the payoff is $C_T = f(S_T) := (S_T - K)^+$ at time T hence equation (2.13.7) yields

$$c_t^C = \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} ((1+u)^k (1+d)^{T-t-k} S_t - K)^+. \tag{2.13.8}$$

Let

$$\gamma_t := \inf\{k \in \mathbb{T} \mid (1+u)^k (1+d)^{T-t-k} S_t - K > 0\}$$

denote the minimum number of up-moves required to be "in the money", that is, to make a strictly positive profit. We can solve γ_t explicitly:

$$\gamma_t = \left\lceil \frac{\ln \frac{K}{(1+d)^{T-t} S_t}}{\ln \frac{1+u}{1+d}} \right\rceil,$$

where ln is the natural logarithm and $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$ denotes the ceiling function of x. Now using the definition of $(\cdot)^+$ we get

$$c_t^C = \frac{1}{(1+r)^{T-t}} \sum_{k=\gamma_t}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} ((1+u)^k (1+d)^{T-t-k} S_t - K). \tag{2.13.9}$$

Let us denote $\psi(T-t, k; q) = \binom{T-t}{k} q^k (1-q)^{T-t-k}$ the probability that a random variable with the binomial distribution of parameters $T-t$ and q takes the value k. We also use the following notation for a summation of those probabilities:

$$\Psi(T-t, \gamma_t; q) = \sum_{k=\gamma_t}^{T-t} \psi(T-t, k; q).$$

Now since $q = (r-d)/(u-d)$, we can write (2.13.9) in a different form

$$c_t^C = S_t \sum_{k=\gamma_t}^{T-t} \binom{T-t}{k} \left(\frac{1+u}{1+r} q\right)^k \Bigg(\underbrace{\frac{1+d}{1+r}(1-q)}_{=1-\frac{1+u}{1+r}q} \Bigg)^{T-t-k} - \frac{K}{(1+r)^{T-t}} \Psi(T-t, \gamma_t; q).$$

Futhermore if we denote $q^* = \dfrac{1+u}{1+r} q$, then we have derived the following result:

Theorem 2.13.10. *(Cox-Ross-Rubinstein pricing formula) The arbitrage-free price of a European call option in the binomial model is given by*

$$c_t^C = S_t \Psi(T-t, \gamma_t; q^*) - \frac{K}{(1+r)^{T-t}} \Psi(T-t, \gamma_t; q) \tag{2.13.11}$$

for $t \in \mathbb{T} \setminus \{T\}$ with notation given above.

Equation (2.13.11) is the discrete-time analogue of the well-known *Black-Scholes pricing formula*. If we add trading times and shrink the lengths of trading periods, we will end up with Black-Scholes formula. See, for example, Section 2.6 in [9] for this approach. Recall that the put-call parity (Proposition 2.7.2) provides us with a way to instantly compute price of a European put option with respect to the price of a call option.

Remark 2.13.12. In the derivation of Theorem 2.13.10 we defined a new constant

$$q^* = \frac{1+u}{1+r} q \in (0,1),$$

where $q = (r-d)/(u-d)$ is associated with the measure $\mathbb{Q}$. It turns out that q^* induces a new measure $\mathbb{Q}^*$ which is also an equivalent martingale measure but with respect to numeraire S. It is clear that S is a martingale with respect to $\mathbb{Q}^*$ since $\widetilde{S}_t = \frac{S_t}{S_t} = 1$, so it suffices to show that $\widetilde{B}$ is also. Straightforward calculation gives

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}(\widetilde{B}_t | \mathcal{F}_{t-1}) &= (1+r)^t \mathbb{E}_{\mathbb{Q}^*}(\frac{1}{S_t} | \mathcal{F}_{t-1}) = \frac{(1+r)^t}{S_{t-1}} \mathbb{E}_{\mathbb{Q}^*}(\frac{1}{1+R_t} | \mathcal{F}_{t-1}) \\
&= \frac{(1+r)^t}{S_{t-1}} \Bigg(\underbrace{\frac{1}{1+u} q^* + \frac{1}{1+d}(1-q^*)}_{=(1+r)^{-1}} \Bigg) = \frac{(1+r)^{t-1}}{S_{t-1}} \\
&= \widetilde{B}_{t-1},
\end{aligned}$$

where we assumed that $R_1, R_2, \ldots, R_t$ are $\mathbb{Q}^*$-independent. △

2.13.3 Hedging

Recall that binomial model is complete hence for every (European) derivative there exist hedging strategy. Let us assume that $\theta = \{(\eta_t, \psi_t) \mid t \in \mathbb{T}\}$ is a hedging strategy for some derivative. We denote the time t value of such portfolio by V_t^θ. Recall that this must coincide with the price of the underlying derivative in arbitrage-free market model. For the sake of clarity we denote shortly

$$V_t^\theta(\omega_t) := V_t^\theta(\omega_1, \ldots, \omega_{t-1}, \omega_t, \omega_{t+1}, \ldots, \omega_T)$$

for $t \in \mathbb{T}$ to emphasize that ω_t is the variable while others remain unchanged. With the corresponding notation we have

$$S_t(\omega_t) = \begin{cases} (1+u)S_{t-1} & \text{for } \omega_t = u \\ (1+d)S_{t-1} & \text{for } \omega_t = d. \end{cases} \tag{2.13.13}$$

This means that

$$
\begin{aligned}
V_t^\theta(\omega_t) &= \eta_t B_t + \psi_t S_t(\omega_t) \\
&= \begin{cases} \eta_t B_t + \psi_t (1+u) S_{t-1} & \text{for } \omega_t = u \\ \eta_t B_t + \psi_t (1+d) S_{t-1} & \text{for } \omega_t = d. \end{cases}
\end{aligned} \tag{2.13.14}
$$

Therefore we have two equation (cases $\omega_t = u$ and $\omega_t = d$) from which we can solve η_t and ψ_t. Solutions are

$$
\psi_t = \frac{V_t^\theta(u) - V_t^\theta(d)}{(u-d) S_{t-1}} \quad \text{and} \quad \eta_t = \frac{(1+u)V_t^\theta(d) - (1+d)V_t^\theta(u)}{(u-d)(1+r)^t}. \tag{2.13.15}
$$

Chapter 3

Continuous-time financial market

In this chapter we assume that the time parameter t (instead of being discrete variable with distinct values $0, 1, 2, \ldots, T$) is continuous variable that can take on all real values in a finite interval $\mathbb{T} = [0, T]$, where $T \in \mathbb{N}$. This seems like a minor change, but it has a significant impact on the complexity of the mathematical theory.

3.1 Brownian motion and semimartingales

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As before, we use the concept of filtration to model the flow of information available to investors. We assume that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, when $s \leq t$ and denote *filtration* (that is, an increasing family of sub-sigma-algebras) by $\mathbb{F} = \{\mathcal{F}_t \,|\, t \in \mathbb{T}\}$. We assume that $\mathbb{F}$ satisfies the usual conditions, that is every null set (a set with zero measure) in $\mathcal{F}$ belongs to $\mathcal{F}_t$ for all $t \in \mathbb{T}$ and filtration is right continuous, that is $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$. Again a stochastic process $X = \{X_t \,|\, t \in \mathbb{T}\}$ is called *adapted* (to the filtration $\mathbb{F}$), if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{T}$.

The next important notion, *Brownian motion*, will be a stochastic component in our continuous-time stock price process. In physics, Brownian motion describes the random movement of a particle in fluid that results from collisions with other particles. Similar movement can be observed in stock prices as prices are driven by "collisions of financial particles" *bids* and *asks*, namely, offers to buy or sell stocks, respectively, made by investors. Analogy is, though, poor since every offer is not immediately executed thus the impact on the price might come with a delay. Furthermore, investors are conscious beings with at least some sort of rational senses, while unperturbed particles are driven by completely unconscious forces (as far as we know).

Definition 3.1.1. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis. An $\mathbb{F}$-adapted process $B = \{B_t \,|\, t \in \mathbb{T}\}$ is a *Brownian motion* if

(i) $\mathbb{P}(B_0 = 0) = 1$;

(ii) for $s \leq t$, a random variable $B_t - B_s$ follows the normal distribution with mean 0 and variance $t - s$;

(iii) for $s \leq t$, a random variable $B_t - B_s$ is independent of $\mathcal{F}_s$;

(iv) the map $t \mapsto B_t(\omega)$ (or *path*) is continuous for almost all $\omega \in \Omega$. ◊

Recall that the real-valued function of *bounded variation* f is such that its *total variation* on a chosen interval is finite. We omit the explicit definition but mention that by total variation, we mean the supremum of sums of differences $|f(x_{i+1}) - f(x_i)|$ taken over the set of all partitions of the interval in question. Now we will introduce some other forms of variation, which we will need later.

Definition 3.1.2. Let $\Pi = \{t_0, t_1, \dots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0, t]$ and let us denote $\|\Pi\| = \max_{k=0,1,\dots,n-1}(t_{k+1} - t_k)$. The *quadratic variation* of stochastic process $X = \{X_t \mid t \in \mathbb{T}\}$ up to time t is denoted by $\langle X \rangle_t$ and defined by

$$\langle X \rangle_t = \operatorname*{plim}_{\|\Pi\| \to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^2$$

assuming that limit exists when it is defined by using *convergence in probability.* More generally the *covariation* of two processes X and Y is denoted and defined by

$$\langle X, Y \rangle_t = \operatorname*{plim}_{\|\Pi\| \to 0} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k}),$$

in particular $\langle X \rangle_t = \langle X, X \rangle_t$. ◊

The Brownian motion is highly studied and its properties are well-known. The next property makes our life both harder and easier in the next section, when we define a stochastic integral with respect to the Brownian motion. We will from now on use character W (as a *Wiener process*) when referring to the Brownian motion for the purpose of distinguishing it later from the non-risky asset B.

Lemma 3.1.3. *Let W be a Brownian motion, then for all $t \in \mathbb{T}$*

$$\langle W \rangle_t = t.$$

Moreover it follows that almost all the paths of the Brownian motion are of unbounded variation on the interval $[0, t]$.

Proof. This result can be found from multiple books which have topics on stochastic calculus. See, for example, the proof of Theorem 3.74 in [15]. Notice, however, that the mentioned proof is for L^2-convergence which then implies convergence in probability. □

Recall the martingale from definition. In continuous time we will generalize this concept.

Definition 3.1.4. A stochastic process $M = \{M_t \mid t \in \mathbb{T}\}$ is called a *martingale* (with respect to $(\mathbb{F}, \mathbb{P})$) if $M_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in \mathbb{T}$ and

$$\mathbb{E}(M_t | \mathcal{F}_s) = M_s \quad \text{for } s \leq t. \tag{3.1.5}$$

Replacing the last equality with $\leq$ ($\geq$) we get *supermartingale* (respectively, *submartingale*.) ◊

Fortunately, Brownian motion is a martingale and it can be used to construct useful martingales which will need later. The next lemma demonstrates this.

Lemma 3.1.6. *Suppose $W = \{W_t \,|\, t \in \mathbb{T}\}$ is a Brownian motion, then W is a martingale (with respect to $(\mathbb{F}, \mathbb{P})$) and process Λ defined by*

$$\Lambda_t = \exp(\sigma W_t - \frac{\sigma^2}{2} t), \quad \sigma \in \mathbb{R}$$

is a martingale (with respect to $(\mathbb{F}, \mathbb{P})$).

Proof. Let us assume that $s \leq t$. Then by the condition (iii) and (ii) of Definition 3.1.1, we have

$$\mathbb{E}(W_t | \mathcal{F}_s) = \mathbb{E}(W_t - W_s + W_s | \mathcal{F}_s) = \underbrace{\mathbb{E}(W_t - W_s)}_{=0} + W_s = W_s.$$

Recall from the basic probability calculus that the moment-generation function of a normally distributed random variable X (with mean ν and variance λ^2) is $M_X(a) = \mathbb{E}(e^{aX}) = \exp(\nu a + \frac{\lambda^2 a^2}{2})$. Now in a similar fashion as before

$$\mathbb{E}(\Lambda_t | \mathcal{F}_s) = \mathbb{E}(e^{\sigma W_t - \frac{\sigma^2}{2} t} | \mathcal{F}_s) = e^{\frac{\sigma^2}{2} t} e^{\sigma W_s} \mathbb{E}(e^{\sigma(W_t - W_s)}) = e^{\sigma W_s} e^{\frac{\sigma^2}{2} t} e^{\frac{(t-s)\sigma^2}{2}} = \Lambda_s.$$

□

Recall that the stopping time $\tau : \Omega \to \mathbb{T}$ is a random variable defined on stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$. Furthermore, the associated stopped process X^τ is defined by $X_t^\tau(\omega) := X_{\tau(\omega) \wedge t}(\omega)$ for $t \in \mathbb{T}$. Next definition generalizes concept of martingale by allowing the localized version of the martingale property.

Definition 3.1.7. A stochastic process $M = \{M_t \mid t \in \mathbb{T}\}$ is called a *local martingale* (with respect to $(\mathbb{F}, \mathbb{P})$) if there exists an increasing sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\mathbb{P}$-almost surely $\lim_{n \to \infty} \tau_n = T$ and stopped process M^{τ_n} is a $(\mathbb{F}, \mathbb{P})$-martingale for every $n \in \mathbb{N}$. Particularly, the following holds $\mathbb{P}$-almost surely

$$\mathbb{E}(M_{s \wedge \tau_n} | \mathcal{F}_t) = M_{t \wedge \tau_n} \quad \text{for } s \leq t. \tag{3.1.8}$$

Sequence $(\tau_n)_{n \in \mathbb{N}}$ is sometimes called a *localization sequence* for M. ♢

All martingales are local martingales (simply consider $\tau_n = T$ for every $n \in \mathbb{N}$) but there exist local martingales that are not martingales. We generalize the concept of martingale even further by introducing the next definition.

Definition 3.1.9. A stochastic process $X = \{X_t \mid t \in \mathbb{T}\}$ is called a *semimartingale* if there exists local martingale $M = \{M_t \mid t \in \mathbb{T}\}$ and process $A = \{A_t \mid t \in \mathbb{T}\}$ which is càdlàg (right-continuous with left limits) and has bounded variation, so that X can be decomposed as

$$X_t = X_0 + M_t + A_t \quad \text{for every } t \in \mathbb{T}. \tag{3.1.10}$$

♢

The semimartingales form a large class of processes including, for example, all càdlàg martingales, submartingales, supermartingales, and later introduced Itô processes. Quadratic variations and covariations can be shown to exist for all processes in this class, and these processes are considered to be "good integrators". Our mission in the next section is to define integration with respect to semimartingales such as Brownian motion.

3.2 Stochastic integrals and Itô calculus

Recall that in discrete time the discounted value of self-financing portfolio was given by

$$\widetilde{V}_t^\theta = V_0^\theta + \sum_{u=1}^{t} \theta_u \cdot \Delta \widetilde{S}_u = V_0^\theta + \sum_{i=0}^{d} \sum_{u=1}^{t} \theta_u^i \Delta \widetilde{S}_u^i,$$

where the discounted price process $\widetilde{S}^i$ is a martingale under the risk-neutral measure. In continuous time we want to extend this representation so the summation (over time) is replaced with integral

$$\int_0^t \theta_u^i d\widetilde{S}_u^i.$$

Now if we assume that $\widetilde{S}^i$ is driven by Brownian motion (which we have proven to be a martingale), we stumble upon a problem with the definition of that integral. In general, integral cannot be defined as $\int_0^t \theta_u^i (\frac{d\widetilde{S}_u^i}{du}) du$ or even as Riemann-Stieltjes integral since we have already stated that almost all sample paths of Brownian motion have unbounded variation and therefore they are not differentiable. In this chapter we will study the aforementioned integral and its properties.

Definition 3.2.1. Stochastic process $X = \{X_t \,|\, t \in \mathbb{T}\}$ is said to be *progressively measurable* with respect to filtration $\mathbb{F} = \{\mathcal{F}_t \,|\, t \in \mathbb{T}\}$ if $X|_{[0,t]}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable for every $t \in \mathbb{T}$. $\diamondsuit$

Progressively measurable processes are also, by definition, adapted to the filtration in question. Inverse does not hold in general but adapted processes with left- or right-continuous paths are progressively measurable.

Definition 3.2.2. Let us assume that stochastic process $H = \{H_t \,|\, t \in \mathbb{T}\}$ is progressively measurable and such that $\int_0^T \mathbb{E}(H_t^2) dt < \infty$. Then we say that process H belongs to the class $\mathbb{L}^2$. $\diamondsuit$

Definition 3.2.3. Process $H \in \mathbb{L}^2$ is called *simple* if there exists random variables e_k, $k = 1, \ldots, N$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(e_{k-1} = e_k) = 0$ for all $k = 2, \ldots, N$ and deterministic points $0 \leq t_0 < t_1 < \ldots < t_N \leq T$ such that

$$H_t = \sum_{k=1}^{N} e_k \mathbb{1}_{(t_{k-1}, t_k]}(t), \quad t \in \mathbb{T}. \tag{3.2.4}$$

$\diamondsuit$

For simple process $H \in \mathbb{L}^2$ with representation (3.2.4) and for Brownian motion W, we define *Itô integral* by

$$\int H_s dW_s := \sum_{k=1}^{N} e_k (W_{t_k} - W_{t_{k-1}}) \tag{3.2.5}$$

also for any $u, t \in \mathbb{T}$ such that $u < t$ we have

$$\int_u^t H_s dW_s = \int H_s \mathbb{1}_{(u,t]}(s) dW_s. \tag{3.2.6}$$

Let us then denote by $\mathcal{M}_c^2$ the *space of continuous martingales* $M = \{M_t \mid t \in \mathbb{T}\}$ such that $M_0 = 0$ and $\|M\|_{\mathbb{T}}^2 := \mathbb{E}(\sup_{t\in\mathbb{T}} |M_t|^2) < \infty$ with $\mathbb{T} = [0, T]$. The normed space $(\mathcal{M}_c^2, \|\cdot\|_{\mathbb{T}})$ is complete (see Lemma 3.43 in [15]) and a process $X = \{X_t \mid t \in \mathbb{T}\}$ defined by Itô integral for simple process $H \in \mathbb{L}^2$, as in (3.2.6),

$$X_t = \int_0^t H_s dW_s, \tag{3.2.7}$$

is continuous martingale with $\|X\|_{\mathbb{T}} < \infty$, that is, $X \in \mathcal{M}_c^2$. For details, see Theorem 4.5 part (5) in [15]. We will not go deeply in to these details of intermediate results as our priorities are eventually elsewhere.

We again state without proving (see Lemma 4.8 in [15] for details) that every $H \in \mathbb{L}^2$ can be approximated by sequence of simple processes $(H^n)_{n\in\mathbb{N}}$ in $\mathbb{L}^2$ so that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T (H_t - H_t^n)^2 dt \right) = 0. \tag{3.2.8}$$

These observations, finally, give us a way to generalize Itô integral for $\mathbb{L}^2$-processes.

Definition 3.2.9. Assume that sequence of simple processes $(H^n)_{n\in\mathbb{N}}$ approximates $H \in \mathbb{L}^2$ as in (3.2.8) then we define the *stochastic integral* of $H \in \mathbb{L}^2$ by

$$\int_0^t H_s dW_s := \lim_{n\to\infty} \int_0^t H_s^n dW_s \tag{3.2.10}$$

This limit is taken in complete space $(\mathcal{M}_c^2, \|\cdot\|_{\mathbb{T}})$ for technical reasons (see discussion in Chapter 4.3 in [15]). $\Diamond$

Next we will review some useful features of such integral.

Lemma 3.2.11. *(Basic properties of stochastic integral) Let us assume that $H, K \in \mathbb{L}^2$ and $0 \le u < v < t \le T$ and further $a, b \in \mathbb{R}$. Then the following properties hold*

(i) linearity: $$\int_0^t (aH_s + bK_s) dW_s = a \int_0^t H_s dW_s + b \int_0^t K_s dW_s, \tag{3.2.12}$$

(ii) additivity: $$\int_u^t H_s dW_s = \int_u^v H_s dW_s + \int_v^t H_s dW_s, \tag{3.2.13}$$

(iii) zero expectation: $$\mathbb{E}\left(\int_u^t H_s dW_s | \mathcal{F}_u \right) = 0, \tag{3.2.14}$$

(iv) Itô isometry: $$\mathbb{E}\left[\left(\int_u^t H_s dW_s \right) \left(\int_u^t K_s dW_s \right) | \mathcal{F}_u \right] = \mathbb{E}\left[\int_u^t H_s K_s ds | \mathcal{F}_u \right]. \tag{3.2.15}$$

Proof. We omit the proofs, but mention that corresponding results can be first proved to the integrals of simple processes (see, for example, Theorem 4.5 in [15]) and then extend to $\mathbb{L}^2$-processes by taking limits. $\square$

Lemma 3.2.16. *(More properties of stochastic integral) Let $H \in \mathbb{L}^2$ and $X = \{X_t \mid t \in \mathbb{T}\}$ be a stochastic process defined by*

$$X_t = \int_0^t H_s dW_s \quad \textit{for every } t \in \mathbb{T}. \tag{3.2.17}$$

Then $X \in \mathcal{M}_c^2$, in particular X is continuous martingale. Furthermore, quadratic variation of X up to time $t \in \mathbb{T}$ is given by

$$\langle X \rangle_t = \int_0^t H_s^2 ds. \tag{3.2.18}$$

Additionally, if H is also deterministic, then X_t is normally distributed with zero expectation and variance given by $\langle X \rangle_t$ for every $t \in \mathbb{T}$.

Proof. We will only demonstrate that, additivity and zero expectation properties of stochastic integral (see Lemma 3.2.11 (ii) and (iii)) yield

$$\mathbb{E}(X_t|\mathcal{F}_u) = X_u + \underbrace{\mathbb{E}\bigg(\int_u^t H_s dW_s|\mathcal{F}_u\bigg)}_{=0} = X_u \tag{3.2.19}$$

for $u < t$. We leave out the rest of the proof. See Theorem 4.5 part (5) and also Propositions 4.24 and 5.13 in [15]. $\square$

If we assume further, that $K \in \mathbb{L}^2$ and Y is another process defined by stochastic integral so that $Y_t := \int_0^t K_s dW_s$, then we can extend (3.2.18) for the covariation of X, Y up to time $t \in \mathbb{T}$ by

$$\langle X, Y \rangle_t = \int_0^t H_s K_s ds. \tag{3.2.20}$$

The next lemma gives us useful tools to deal with covariation of two processes. Notice that these hold for processes defined by stochastic integral as in (3.2.32), since Lemma 3.2.16 states that such processes belong to $\mathcal{M}_c^2$.

Lemma 3.2.21. *(Properties of covariation) Let us assume that $X, Y, Z \in \mathcal{M}_c^2$ and A, B are continuous processes with bounded variation and $a, b \in \mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle_t$ the covariation of two underlying processes up to time $t \in \mathbb{T}$. Then, for every $t \in \mathbb{T}$, the following identities hold:*

$$\textit{(i) commutativity:} \quad \langle X, Y \rangle_t = \langle Y, X \rangle_t, \tag{3.2.22}$$

$$\textit{(ii) bilinearity:} \quad \langle aX + bY, Z \rangle_t = a\langle X, Z \rangle_t + b\langle Y, Z \rangle_t, \tag{3.2.23}$$

$$\textit{(iii) polarization:} \quad 4\langle X, Y \rangle_t = \langle X + Y \rangle_t - \langle X - Y \rangle_t, \tag{3.2.24}$$

$$\textit{(iv) null contribution:} \quad \langle X + A, Y + B \rangle_t = \langle X, Y \rangle_t. \tag{3.2.25}$$

Furthermore, the process defined by $\langle X, Y \rangle_t$ is unique continuous process with bounded variation and $\langle X, Y \rangle_0 = 0$ almost surely.

Proof. See Section 4.3.5 in [15]. □

Now we want to generalize earlier ideas even further.

Definition 3.2.26. Let us assume and that stochastic process $H = \{H_t \,|\, t \in \mathbb{T}\}$ is progressively measurable and such that $\int_0^T |H_t|^p dt < \infty$ almost surely. Then we say that process H belongs to the class $\mathbb{L}^p_{\text{loc}}$ for some $p \geq 1$. ♢

Definition 3.2.27. Let us assume that $H \in \mathbb{L}^2_{\text{loc}}$ and define increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n = \inf\{t \in \mathbb{T} \mid A_t \geq n\} \wedge T \quad \text{with} \quad A_t = \int_0^t H_s^2 ds \quad \text{for } t \in \mathbb{T}. \tag{3.2.28}$$

Let us define further process H^n by $H_t^n := H_t \mathbb{1}_{\{t\leq\tau_n\}} \in \mathbb{L}^2$ for every $n \in \mathbb{N}$ and $t \in \mathbb{T}$. We define stochastic integral of $H \in \mathbb{L}^2_{\text{loc}}$ (see also required Definition 3.2.9 for the stochastic integral of H^n) by

$$\int_0^t H_s dW_s := \lim_{n\to\infty} \int_0^t H_s^n dW_s. \tag{3.2.29}$$

Here limit exists almost surely. ♢

Previous definition makes a lot of assumptions. Process H^n, indeed, belongs to $\mathbb{L}^2$ for every $n \in \mathbb{N}$, since

$$\mathbb{E}\left(\int_0^T (H_t^n)^2 dt\right) = \mathbb{E}\left(\int_0^{\tau_n} H_t^2 dt\right) = \mathbb{E}(A_{\tau_n}) \leq n. \tag{3.2.30}$$

Thus, by Lemma 3.2.16, H^n is also continuous martingale. Also the sequence $(\tau_n)_{n\in\mathbb{N}}$ is, indeed, increasing sequence of stopping times and further $\tau_n \to T$ almost surely as $n \to \infty$. See Section 4.4 in [15] for more information about this definition.

When proceeding to stochastic integrals of $\mathbb{L}^2_{\text{loc}}$-processes, we lose some of the properties we had in $\mathbb{L}^2$-case. For example, the process defined by stochastic integral of $\mathbb{L}^2_{\text{loc}}$-process is not generally a martingale anymore. Fortunately, the mentioned process is a local martingale (recall Definition 3.1.7).

Lemma 3.2.31. *Let us assume that $H \in \mathbb{L}^2_{\text{loc}}$ and that stochastic process $X = \{X_t \mid t \in \mathbb{T}\}$ is defined by*

$$X_t = \int_0^t H_s dW_s \quad \textit{for every } t \in \mathbb{T}. \tag{3.2.32}$$

Then X is a continuous (paths are almost surely continuous) ***local*** *martingale with localization sequence $(\tau_n)_{n\in\mathbb{N}}$ and quadratic variation $\langle X\rangle_t = A_t$ defined by (3.2.28). Furthermore, the process X is a (true) martingale if the following condition holds*

$$\mathbb{E}\left[\left(\int_0^T H_t^2 dt\right)^{\frac{1}{2}}\right] < \infty. \tag{3.2.33}$$

Proof. We will omit the proof but mention that the last claim follows from two rather well-known results called the *Hölder's inequality* and the *Burkholder-Davis-Gundy's inequality*. See Theorem 4.42, Proposition 4.43 and Corollary 4.48 from [15] for details. □

Remark 3.2.34. (Extensions, when shifting from $\mathbb{L}^2$ to $\mathbb{L}^2_{\text{loc}}$) Notice that the formula (3.2.20) holds likewise, if (instead of assuming $H, K \in \mathbb{L}^2$) we assume that $H, K \in \mathbb{L}^2_{\text{loc}}$. Furthermore, the properties of covariation given by Lemma 3.2.21 apply if we replace $\mathcal{M}^2_c$ with the space of continuous local martingales $\mathcal{M}_{\text{c,loc}}$. Thus aforementioned properties can be used for processes defined by stochastic integrals of $H, K \in \mathbb{L}^2_{\text{loc}}$, since these are local martingale, as previous lemma suggests. △

Now we are prepared to introduce a very important notion.

Definition 3.2.35. Consider stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and Brownian motion W. A process $X = \{X_t \mid t \in \mathbb{T}\}$ of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \quad \text{for every } t \in \mathbb{T} \tag{3.2.36}$$

is called *Itô process*, if X_0 is $\mathcal{F}_0$-measurable random variable (thus constant, if $\mathcal{F}_0$ is trivial), the processes $\mu = \{\mu_t \,|\, t \in \mathbb{T}\} \in \mathbb{L}^1_{\text{loc}}$ and $\sigma = \{\sigma_t \,|\, t \in \mathbb{T}\} \in \mathbb{L}^2_{\text{loc}}$, that is, μ and σ are progressively measurable with

$$\int_0^T |\mu_t| dt < \infty \quad \text{and} \quad \int_0^T (\sigma_t)^2 dt < \infty$$

almost surely. ◊

We often write (3.2.36) more shortly with notation

$$dX_t = \mu_t dt + \sigma_t dW_t. \tag{3.2.37}$$

This presentation is informally referred to as *dynamics*. Here the process μ is called *drift* coefficient and the process σ is called *diffusion* coefficient. Itô process is unique in a such way that if X has also representation (along with representation (3.2.37))

$$dX_t = \widehat{\mu}_t dt + \widehat{\sigma}_t dW_t,$$

then $\mu_t = \widehat{\mu}_t$ and $\sigma_t = \widehat{\sigma}_t$ almost surely. Next lemma will show that drift coefficient determines whether the Itô process is local martingale, and that only the diffusion coefficient will contribute to quadratic variation of an Itô process.

Lemma 3.2.38. *Suppose that X is an Itô process as in (3.2.36), then X is a local martingale if and only if $\mu_t = 0$ for every $t \in \mathbb{T}$ almost surely. Furthermore, the quadratic variation of X up to time $t \in \mathbb{T}$ is given by*

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds \quad \textit{or more shortly} \quad d\langle X \rangle_t = \sigma_t^2 dt. \tag{3.2.39}$$

Proof. See Remark 4.5 in [15] and for the quadratic variation Remark 3.2.34. □

Proposition 3.2.40. *(Multidimensional Itô's formula) Suppose $X^1, \dots, X^d$ are Itô processes ($d \in \mathbb{N}$) with continuous paths, $X_t = (X_t^1, \dots, X_t^d)$, and $f : \mathbb{R}^d \to \mathbb{R}$ is continuous function with continuous first two derivatives. Then for all $t \in \mathbb{T}$*

$$f(X_t) = f(X_0) + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_s)dX_s^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)d\langle X^i, X^j\rangle_s \tag{3.2.41}$$

or more shortly

$$df(X_t) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t)d\langle X^i, X^j\rangle_t.$$

Proof. See Chapter 11 in [1] for proof concerning continuous semimartingales. We will mention, though, that if we denote by decomposition $X_t = X_0 + M_t + A_t$ of continuous semimartingale where M_t is local martingale part and A_t is part with bounded variation, then Itô process, X as in (3.2.36), is semimartingale with

$$M_t = \int_0^t \sigma_s dW_s \quad \text{and} \quad A_t = \int_0^t \mu_s ds. \tag{3.2.42}$$

□

In the one-dimensional case ($d = 1$) proposition above would yield clean dynamics

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t. \tag{3.2.43}$$

Corollary 3.2.44. *(Integration by parts formula) Let X and Y be Itô processes with continuous paths. Then we have*

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t \tag{3.2.45}$$

or more shortly

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t.$$

Proof. Consider multidimensional Itô's formula (Proposition 3.2.40) for function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$ or see Corollary 11.3 in [1]. □

Another frequently used form of Itô's formula is given in the next corollary. We will omit the proof.

Corollary 3.2.46. *(Itô's lemma) If $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous function and continuously differentiable in the first component and twice continuously differentiable in the second, then*

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)d\langle X\rangle_t \tag{3.2.47}$$

for Itô process X and particularly

$$df(t, W_t) = \frac{\partial f}{\partial t}(t, W_t)dt + \frac{\partial f}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, W_t)dt. \tag{3.2.48}$$

Example 3.2.49. Consider process $S = \{S_t \,|\, t \in \mathbb{T}\}$ defined by

$$S_t = f(t, W_t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t},$$

where W is a Brownian motion and $S_0, \mu, \sigma \in \mathbb{R}$ are constants. By Itô's lemma (Corollary 3.2.46), we get

$$dS_t = (\mu - \frac{\sigma^2}{2})S_t dt + S_t \sigma dW_t + \frac{1}{2}\sigma^2 S_t dt = \mu S_t dt + S_t \sigma dW_t.$$

As we deduced, the process S satisfies the stochastic differential equation $dS_t = S_t(\mu dt + \sigma dW_t)$ and it is called a *geometric Brownian motion*. It plays a key role in the Black-Scholes model as it is used to model stock prices.

On the other way around, if we only knew the dynamics of process S to be $dS_t = \mu S_t dt + \sigma S_t dW_t$, we could solve S explicit by applying one-dimensional Itô's formula (3.2.43) to function $f(x) = \ln(x)$. Now

$$\begin{aligned} d(\ln(S_t)) &= \frac{1}{S_t} dS_t + \frac{1}{2} \cdot \Big(- \frac{1}{S_t^2} \Big) d\langle S \rangle_t \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \end{aligned}$$

And by integrating we get

$$\ln(S_t) = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \quad \iff \quad S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}.$$

We also notice that $\ln(S_t)$ is normally distributed (since W_t is normally distributed) with

$$\mathbb{E}(\ln(S_t)) = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t \quad \text{and} \quad \mathrm{Var}(\ln(S_t)) = \sigma^2 t.$$

△

Remark 3.2.50. (Extensions of stochastic integral) Earlier, we only considered Brownian motion W as an integrator in stochastic integrals. We will now review, very briefly, how the integrator can be eventually replaced with semimartingale to generalize ideas. More comprehensive construction can be found from Chapter 10 of [1].

(i) Let us assume M is square integrable martingale with continuous paths. Suppose further that H is predictable process with $\int_0^T \mathbb{E}(H_t^2) d\langle M \rangle_t < \infty$ (analogical with conditions of $\mathbb{L}^2$-processes in Brownian motion case). As before, we can approximate H with a sequence of simple processes $(H^n)_{n\in\mathbb{N}}$ that have representation

$$H_t^n = \sum_{k=1}^{N} e_k^n \mathbb{1}_{(t_{k-1}, t_k]}(t) \tag{3.2.51}$$

with e_k^n bounded and $\mathcal{F}_{t_{k-1}}$-measurable random variables and for which we define

$$\int_0^t H_s^n dM_s := \sum_{k=1}^{N} e_k^n (M_{t \wedge t_k} - M_{t \wedge t_{k-1}}). \tag{3.2.52}$$

If aforementioned sequence $(H^n)_{n\in\mathbb{N}}$ approximates H so that $\mathbb{E}[\int_0^T (H_t - H_t^n)^2 d\langle M\rangle_t] \to 0$ as $n \to \infty$, then we define stochastic integral

$$X_t = \int_0^t H_s dM_s \tag{3.2.53}$$

as the limit of integrals $\int_0^t H_s^n dM_s$ with respect to norm $\|\cdot\|_{\mathbb{T}}$ (see the beginning of the section). This process X, defined by stochastic integral, is a continuous martingale with

$$\langle X\rangle_t = \int_0^t H_s^2 d\langle M\rangle_s. \tag{3.2.54}$$

Suppose now that predictable H only satisfies $\int_0^T H_t^2 d\langle M\rangle_t < \infty$ almost surely (analogical with conditions of $\mathbb{L}^2_{\text{loc}}$-processes in Brownian motion case), then we define τ_n by (3.2.28) but replace integral with corresponding $A_t = \int_0^t H_s^2 d\langle M\rangle_s$. Since $M_{t\wedge\tau_n}$ is square integrable martingale, we set $\int_0^t H_s dM_s := \int_0^t H_s dM_{s\wedge\tau_n}$.

(ii) Let us assume that M is a continuous local martingale. We denote $\lambda_n = \inf\{t \in \mathbb{T} : |M_t| \geq n\}$ and state without proving that a stopped process defined by $M_{t\wedge\lambda_n}$ is square integrable martingale. For the H as in part (i), we define

$$\int_0^t H_s dM_s := \int_0^t H_s dM_{s\wedge\lambda_n}. \tag{3.2.55}$$

(iii) Let us assume that X is a semimartingale with continuous paths and $X_t = M_t + A_t$ where M is a local martingale and A is a process of bounded variation. Suppose further that $\int_0^T H_t^2 d\langle M\rangle_t + \int_0^T |H_t||dA_t| < \infty$, where $|dA_t|$ denotes the total variation differential of A. Then we set, separately,

$$\int_0^t H_s dX_s := \int_0^t H_s dM_s + \int_0^t H_s dA_s, \tag{3.2.56}$$

where the last integral is Lebesgue-Stieltjes integral. Notice that Itô's formula (Proposition 3.2.40) holds, if we replace words "Itô processes" with "semimartingales". △

3.3 Girsanov's theorem

Recall that it was important in discrete-time models to find a so-called risk neutral measure such that the discounted asset price process was martingale under that measure. This section gives tools to overcome this problem in continuous time.

Definition 3.3.1. Suppose $\lambda = \{\lambda_t \,|\, t \in \mathbb{T}\} \in \mathbb{L}^2_{\text{loc}}$, that is, λ is progressively measurable process such that $\int_0^T \lambda_s^2 ds < \infty$ almost surely. We call process $\Lambda = \{\Lambda_t \,|\, t \in \mathbb{T}\}$ *stochastic exponential* (associated to λ) if

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds \right) \quad \forall t \in \mathbb{T}, \tag{3.3.2}$$

where W is a Brownian motion under $\mathbb{P}$. ♢

Recall that stochastic exponential Λ associated to λ is a martingale, at least when λ is real-valued constant (see Lemma 3.1.6). Actually it suffices that λ is bounded process, which is useful in many financial applications. Even more general condition, such that Λ is still martingale, is called *Novikov's condition*, which is defined by

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \lambda_s^2 ds\right)\right] < \infty.$$

If this is satisfied together with other conditions, the next theorem will show how we can eliminate the drift from the Itô process by changing probability measure and Brownian motion

Theorem 3.3.3. *(Girsanov's theorem) Suppose $\lambda = \{\lambda_t \,|\, t \in \mathbb{T}\} \in \mathbb{L}^2_{loc}$, that is, λ is progressively measurable process such that $\int_0^T \lambda_s^2 ds < \infty$ almost surely. If the stochastic exponential Λ associated to λ is a martingale (with respect to $(\mathbb{F}, \mathbb{P})$), then a new measure $\mathbb{Q}$ (restricted to $\mathcal{F}_T$) defined by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \Lambda_T = \exp\left(-\int_0^T \lambda_s dW_s - \frac{1}{2}\int_0^T \lambda_s^2 ds\right)$$

is such that the process $W^{\mathbb{Q}}$ given by

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_s ds \quad t \in \mathbb{T}, \tag{3.3.4}$$

is a Brownian motion under $\mathbb{Q}$.

Proof. See, for example, the proof of Theorem 7.2.3 in [9]. □

The process λ in the Girsanov's theorem is called the *Girsanov's kernel*. The theorem can be used when we need to solve an asset's or rate's dynamics under a new probability measure. Thus it is sometimes useful to write (3.3.4) more shortly as

$$dW_t^{\mathbb{Q}} = dW_t + \lambda_t dt. \tag{3.3.5}$$

Example 3.3.6. Consider Itô processes $S = \{S_t \,|\, t \in \mathbb{T}\}$ and $B = \{B_t \,|\, t \in \mathbb{T}\}$ with $\mathbb{P}$-dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt, \end{aligned}$$

where $\mu, \sigma, r \in \mathbb{R}$ are constants and W is a Brownian motion under $\mathbb{P}$. Let us define a new process $\widetilde{S}$ by $\widetilde{S}_t = S_t / B_t$ for all $t \in \mathbb{T}$. Integration by parts formula (Corollary 3.2.44) yields

$$\begin{aligned} d\widetilde{S}_t &= d(S_t B_t^{-1}) \\ &= B_t^{-1} dS_t + S_t d(B_t^{-1}) \\ &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t \end{aligned} \tag{3.3.7}$$

This can be written as

$$d\widetilde{S}_t = \sigma\widetilde{S}_t\Big(\frac{\mu - r}{\sigma}dt + dW_t\Big). \tag{3.3.8}$$

Now by choosing the Girsanov's kernel as constant

$$\lambda_t := \frac{\mu - r}{\sigma} \quad \forall t \in \mathbb{T}, \tag{3.3.9}$$

we can use Girsanov's theorem 3.3.3 to get rid of the drift term in (3.3.8) and the dynamics of $\widetilde{S}$ become

$$d\widetilde{S}_t = \sigma\widetilde{S}_t dW_t^{\mathbb{Q}}, \tag{3.3.10}$$

where $W^{\mathbb{Q}}$ is a Brownian motion under measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(- \Big[\frac{\mu - r}{\sigma}\Big]W_T - \frac{1}{2}\Big[\frac{\mu - r}{\sigma}\Big]^2 T\Big). \tag{3.3.11}$$

$\triangle$

3.4 Feynman-Kac formula

Consider the following boundary value problem. We are given functions $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$ and constant $r \in \mathbb{R}$ and we have to find function $V : [0, T] \times \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\partial V}{\partial t}(t, x) + \mu(t, x)\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) = 0 \tag{3.4.1}$$

$$V(T, x) = F(x). \tag{3.4.2}$$

To shorten the notation, let us introduce differential operator $\mathcal{L}$ defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \mu(t, x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2}{\partial x^2} - r, \tag{3.4.3}$$

so that (3.4.1) is just $\mathcal{L}[V(t, x)] = 0$. Instead of trying to solve the problem directly, we drag stochastics into aboard. First let us assume that some solution V indeed exists and that stochastic process $X = \{X_s \,|\, s \in [t, T]\}$ solves stochastic differential equation

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \tag{3.4.4}$$

$$X_t = x, \tag{3.4.5}$$

for fixed $t \in \mathbb{T}$ and $x \in (0, \infty)$. Now let us apply Itô's lemma (Corollary 3.2.46) to process $V(s, X_s)$. That yields

$$\begin{aligned} dV(s, X_s) &= \frac{\partial V}{\partial s}(s, X_s)ds + \frac{\partial V}{\partial x}(s, X_s)dX_s + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(s, X_s)d\langle X\rangle_s \\ &= \Big(\frac{\partial V}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial V}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 V}{\partial x^2}(s, X_s)\Big)ds \\ &+ \sigma(s, X_s)\frac{\partial V}{\partial s}(s, X_s)dW_s \\ &= \big(\mathcal{L}[V(s, X_s)] + rV(s, X_s)\big)ds + \sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)dW_s \end{aligned} \tag{3.4.6}$$

for $s \in [t, T]$. Then let us apply integration by parts formula (Proposition 3.2.44) to the "discounted" process $e^{-rs}V(s, X_s)$. We get

$$\begin{aligned} d(e^{-rs}V(s, X_s)) &= -re^{-rs}V(s, X_s)ds + e^{-rs}dV(s, X_s) \\ &= e^{-rs}\mathcal{L}[V(s, X_s)]ds + e^{-rs}\sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)dW_s \end{aligned} \tag{3.4.7}$$

Recall that $\mathcal{L}[V(t, x)] = 0$, so the drift term vanishes and we have

$$e^{-rT}V(T, X_T) = e^{-rt}V(t, X_t) + \int_t^T e^{-rs}\sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)dW_s. \tag{3.4.8}$$

Now taking the expected values of both sides yields

$$\begin{aligned} V(t, x) &= \mathbb{E}(e^{-r(T-t)}V(T, X_T) \,|\, X_t = x) \\ &+ \mathbb{E}\Big(\int_t^T e^{-r(s-t)}\sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)dW_s \,|\, X_t = x\Big). \end{aligned} \tag{3.4.9}$$

Assuming convenient integrability conditions on $\sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)$ and recalling that on the boundary we have $V(T, x) = F(x)$, gives us solution

$$V(t, x) = \mathbb{E}(e^{-r(T-t)}F(X_T) \,|\, X_t = x). \tag{3.4.10}$$

Let us sum up the the result.

Proposition 3.4.11. *(Feynman-Kac formula) Assume that V is a solution to problem given in (3.4.1)-(3.4.2), that is, on the boundary $V(T, x) = F(x)$ and otherwise*

$$\frac{\partial V}{\partial t}(t, x) + \mu(t, x)\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) = 0$$

Suppose that X satisfies (3.4.4)-(3.4.5), namely,

$$X_t = x \quad \text{and} \quad dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \quad \text{for } s \in (t, T] \tag{3.4.12}$$

Assume further that

$$\int_0^t \mathbb{E}\Big[\Big(\sigma(s, X_s)\frac{\partial V}{\partial x}(s, X_s)\Big)^2\Big]ds < \infty \quad \forall t \in \mathbb{T}.$$

Then V has the representation

$$V(t, x) = \mathbb{E}(e^{-r(T-t)}F(X_T) \,|\, X_t = x). \tag{3.4.13}$$

3.5 Pricing and free lunch

We start by reviewing some of familiar concepts from discrete-time financial markets. Consider financial market with one riskless asset S^0 and risky assets $S^1, \dots S^d$. We denote by $S_t = (S_t^0, S_t^1, \dots, S_t^d)$ the vector of asset prices at time $t \in \mathbb{T}$. A *strategy* (or a *portfolio*) is process $\theta = \{\theta_t \,|\, t \in \mathbb{T}\} = \{(\theta_t^0, \theta_t^1, \dots, \theta_t^d) \,|\, t \in \mathbb{T}\}$ with values in $\mathbb{R}^{d+1}$, where θ_t^0 denotes the amount of riskless asset held at time $t \in \mathbb{T}$ and θ_t^i denotes the amount of risky asset i held at time $t \in \mathbb{T}$ for $i = 1, \dots, d$. The *value of the portfolio* θ at time $t \in \mathbb{T}$ is then given by

$$V_t^\theta = \theta_t \cdot S_t := \sum_{i=0}^{d} \theta_t^i S_t^i. \tag{3.5.1}$$

Recall that in discrete-time (see equation (2.3.6)) self-financing strategy satisfied the following identity

$$\Delta V_t^\theta = \theta_t \cdot \Delta S_t = \sum_{i=0}^{d} \theta_t^i \Delta S_t^i.$$

The continuous-time analogue of this condition is received by replacing the discrete-time difference Δ with continuous-time counterpart. In order to implement this replacement, we need to make some assumptions about the processes S^i. Let us assume that $S_0^0 = 1$ and

$$S_t^0 = \exp\left(\int_0^t r_u du\right) \quad \text{for } t \in \mathbb{T}, \tag{3.5.2}$$

where process r satisfies $\int_0^T |r_u| du < \infty$ almost surely. Now dynamics of S^0 are given by $dS_t^0 = r_t S_t^0 dt$. For risky asset i, we assume that there exists Itô process X^i such that

$$S_t^i = e^{X_t^i} \quad \text{with} \quad dX_t^i = b_t^i dt + a_t^i dW_t^i \tag{3.5.3}$$

for $t \in \mathbb{T}$ and $i = 1, \dots d$. Furthermore, we assume that there exists some positive constant $C \in \mathbb{R}$ such that (almost surely)

$$\int_0^T r_u^2 du + \sum_{i=1}^{d} \int_0^T (a_u^i)^2 du \leq C.$$

In general case, we would demand for the price processes of risky assets only to be positive semimartingales. That is the starting point in [12], which leads to mathematically fairly demanding arbitrage theory. We return to defining self-financing strategies by also requiring some condition for components of θ and give the following definition:

Definition 3.5.4. Assume that the components $\theta^0, \theta^1, \dots, \theta^d$ are progressively measurable processes. A strategy $\theta = \{(\theta_t^0, \theta_t^1, \dots, \theta_t^d) \,|\, t \in \mathbb{T}\}$ is *self-financing* if the following conditions hold (almost surely)

$$\int_0^T |\theta_t^0| dt < \infty \quad \text{and} \quad \int_0^T (\theta_t^i)^2 dt < \infty \quad \text{for each } i = 1, \dots, d, \tag{3.5.5}$$

that is, $\theta^0 \in \mathbb{L}^1_{\text{loc}}$ and $\theta^1, \dots, \theta^d \in \mathbb{L}^2_{\text{loc}}$, and further the value of the portfolio satisfies dynamics

$$dV_t^\theta = \theta_t \cdot dS_t = \sum_{i=0}^{d} \theta_t^i dS_t^i. \tag{3.5.6}$$

$\diamondsuit$

Let us choose S^0 as numeraire (see Section 2.3) and use again notation $\widetilde{V}_t^\theta := V_t^\theta / S_t^0$ for the discounted value of portfolio θ and $\widetilde{S}_t^i := S_t^i / S_t^0$ for the discounted price process of asset i at time $t \in \mathbb{T}$. Then we have the next useful proposition

Proposition 3.5.7. *Let us assume that strategy θ with progressively measurable components satisfies conditions (3.5.5). Then θ is self-financing if and only if*

$$\widetilde{V}_t^\theta = V_0^\theta + \sum_{i=1}^{d} \int_0^t \theta_u^i d\widetilde{S}_u^i \tag{3.5.8}$$

Proof. "$\Rightarrow$": We start by using integration by parts formula (Corollary 3.2.44) and drop off the crossvariation term since it is zero. Hence discounted value dynamics are

$$d\widetilde{V}_t^\theta = d((S_t^0)^{-1} V_t^\theta) = V_t^\theta d((S_t^0)^{-1}) + (S_t^0)^{-1} dV_t^\theta. \tag{3.5.9}$$

We continue by putting the sum form of V_t^θ and self-financing dynamics dV_t^θ (3.5.6) in to previous equation:

$$d\widetilde{V}_t^\theta = \sum_{i=0}^{d} \theta_t^i S_t^i d((S_t^0)^{-1}) + (S_t^0)^{-1} \sum_{i=0}^{d} \theta_t^i dS_t^i = \sum_{i=0}^{d} \theta_t^i \underbrace{[S_t^i d((S_t^0)^{-1}) + (S_t^0)^{-1} dS_t^i]}_{= d\widetilde{S}_t^i}. \tag{3.5.10}$$

By noticing that $d\widetilde{S}_t^0 = 0$, since $\widetilde{S}_t^0 = S_t^0 / S_t^0 = 1$, we get

$$d\widetilde{V}_t^\theta = \sum_{i=1}^{d} \theta_t^i d\widetilde{S}_t^i. \tag{3.5.11}$$

This is shortly written (3.5.5). For the inverse "$\Leftarrow$", we use the same arguments but start by writing $dV_t^\theta = d(S_t^0 \widetilde{V}_t^\theta)$ and, conversely, insert discounted counterparts $\widetilde{V}_t^\theta$ and $d\widetilde{V}_t^\theta$. $\square$

Let us continue by assuming that W is a d-dimensional Brownian motion, where components $W^1, \dots, W^d$ come from diffusions of risky assets as in (3.5.3). We assume that W has a constant correlation matrix. With the assumptions we have made in this section, we can price European derivatives with an equivalent martingale measure in a similar fashion as we did in discrete time. Let us elaborate this.

Definition 3.5.12. We call the probability measure $\mathbb{Q}$ an *equivalent martingale measure* with numeraire S^0, if $\mathbb{Q} \sim \mathbb{P}$ and the process of discounted prices defined by

$$\widetilde{S}_t := \frac{S_t}{S^0} = S_t \exp\left(- \int_0^t r_u du \right) \tag{3.5.13}$$

is a (true) martingale under $\mathbb{Q}$. $\Diamond$

The existence of such measure is determined by the so-called *market price of risk*. We denote this by d-dimensional process λ and the condition for existence by

$$\int_0^T |\lambda_t^i|^2 dt < \infty, \quad \text{where} \quad \lambda_t^i = \frac{b_t^i - r_t}{a_t^i} \tag{3.5.14}$$

for every $i = 1, \dots, d$ and $t \in \mathbb{T}$. This condition comes fundamentally from demands for the Girsanov's kernel in the Girsanov's theorem, even though we did not represent the multidimensional version of aforementioned theorem. In our special case, we also have:

Proposition 3.5.15. *Let us assume that θ is a self-financing strategy and $\mathbb{Q}$ is an equivalent martingale measure. Then the process of discounted value $\widetilde{V}_t^\theta := V_t^\theta / S_t^0$ is a (true) martingale under $\mathbb{Q}$, if $\theta^i a^i \in \mathbb{L}^2(\mathbb{P})$ for all $i = 1, \dots, d$. Particularly, we have*

$$V_t^\theta = \mathbb{E}_{\mathbb{Q}}\left[V_T \exp\left(- \int_t^T r_u du \right) | \mathcal{F}_t^W \right] \quad \textit{for every } t \in \mathbb{T}. \tag{3.5.16}$$

Proof. See the proof of Proposition 10.41 in [15]. $\square$

In this context we say that strategy θ is *admissible*, if the discounted value process $\widetilde{V}^\theta$ is a (true) martingale under every equivalent martingale measure. We use this definition in the next one.

Definition 3.5.17. A European derivative X is such that its payoff X_T is $\mathcal{F}_T^W$-measurable random variable with $X_T \in L^p(\mathbb{P})$ for some $p > 1$. A derivative X is called *replicable*, if there exists a self-financing admissible strategy θ such that $V_T^\theta = X_T$ almost surely. Then θ is called the *replicating strategy* of derivative X. $\Diamond$

For European derivative X, we say that its *risk-neutral price* at time t with respect to equivalent martingale measure $\mathbb{Q}$ is given by

$$c_t^X(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}\left[X_T \exp\left(- \int_t^T r_u du \right) | \mathcal{F}_t^W \right]. \tag{3.5.18}$$

If the X is additionally replicable, then the risk-neutral price is given by the value of replicating strategy which is determined by equivalent martingale measure (assuming admissibility). In this instance, every replicating strategy and equivalent martingale measure agree with the same price. The next theorem states this important result.

Theorem 3.5.19. *Let us assume that European derivative X is replicable. Then for every replicating strategy θ and for every equivalent martingale measure $\mathbb{Q}$, we have that the risk-neutral price of X is given by*

$$c_t^X := V_t^\theta = \mathbb{E}_{\mathbb{Q}}\Big[X_T \exp\Big(-\int_t^T r_u du\Big)|\mathcal{F}_t^W\Big]. \tag{3.5.20}$$

Proof. See the proof of Proposition 10.49 in [15]. □

Actually in our special case, if we assume that equivalent martingale exists, then every European derivative is replicable (recall *completeness*) and the equivalent measure is unique. This is consequence from the assumption that the number of risky assets is equal to dimension of the Brownian motion W. This statement is studied more thoroughly in Section 10.2.6 in [15].

In the earlier discussion, we avoided speaking about the arbitrage. The reason is that in the continuous time the concept of arbitrage gets more complicated at least in a general model, where the price processes of risky assets are assumed to be semimartingales. In that case we consider so-called *no free lunch with vanishing risk* condition instead of definition of arbitrage we got used to in discrete time. We will not give the definition for aforementioned condition, but it can be found from Chapter 8 of [7].

Without going to details, we present the continuous-time fundamental theorem of asset pricing. In the financial market model, where S is R^d-valued semi-martingale, the following two conditions are equivalent. The first one says: there exists probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S is a *sigma-martingale* under $\mathbb{Q}$. The second one states: the process S satisfies condition of no free lunch with vanishing risk.

The sigma-martingale is a semi-martingale for which there exists positive predictable process such that the stochastic integral of this process with respect to semi-martingale in question is a martingale. If we assume that S in abovesaid theorem is also locally bounded, then we can replace the word "sigma-martingale" with "local martingale". The proof of the theorem is far from trivial. An interested reader is directed to [7], which contains an introduction and original papers published in this topic.

3.6 Black-Scholes model

In previous sections we constructed methods to survive in the continuous-time setting. In this section we are going to use these methods for pricing in the Black-Scholes model. Actually we have already used some of these methods in the Black-Scholes framework in Examples 3.2.49 and 3.3.6. The Black-Scholes model originates from papers [4] published by Black and Scholes and [13] published by Merton in 1973. Merton and Scholes received the 1997 Nobel Prize in Economics for their work. Black was not awarded the prize due to his death in 1995.

3.6.1 Dynamics

In the Black-Scholes model, there is one riskless asset B and one risky asset S. Riskless asset has deterministic dynamics

$$dB_t = rB_t dt. \tag{3.6.1}$$

where $r \in \mathbb{R}$ is constant. We assume that $B_0 = 1$ therefore we have explicitly

$$B_t = e^{rt}. \tag{3.6.2}$$

The dynamics of the risky asset given by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3.6.3}$$

where $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are constants and W is a Brownian motion under probability measure $\mathbb{P}$. We also assume that S_0 is constant, therefore recalling Example 3.2.49, we have by Itô's formula

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \tag{3.6.4}$$

Let us choose B as a numeraire and define discounted stock process $\widetilde{S}$ by $\widetilde{S}_t = S_t/B_t$ for all $t \in \mathbb{T}$. Recall that in Example 3.3.6 we used Girsanov's theorem to obtain the dynamics

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dW_t^{\mathbb{Q}} \tag{3.6.5}$$

where $W^{\mathbb{Q}}$ is a Brownian motion under probability measure $\mathbb{Q}$ defined in (3.3.11). So by Itô's formula

$$\widetilde{S}_t = S_0 \exp\left(\sigma W_t^{\mathbb{Q}} - \frac{\sigma^2}{2}t\right), \tag{3.6.6}$$

hence Lemma 3.1.6 states that $\widetilde{S}$ is a $\mathbb{Q}$-martingale. Using the integration by parts formula (Corollary 3.2.44) and equation (3.6.5), we also observe that

$$\begin{aligned} dS_t &= d(\widetilde{S}_t B_t) \\ &= B_t d\widetilde{S}_t + \widetilde{S}_t dB_t \\ &= rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \end{aligned} \tag{3.6.7}$$

so loosely speaking the drift of the stock process has declined to the level of riskless asset's drift under the change of measure. This suggests that measure $\mathbb{Q}$ is neutral towards risk taking.

3.6.2 Black-Scholes partial differential equation

Definition 3.6.8. Strategy $\theta = \{(\eta_t, \psi_t) \,|\, t \in \mathbb{T}\}$ is called *Markovian* if for every $t \in \mathbb{T}$

$$\eta_t = \eta(t, S_t) \quad \text{and} \quad \psi_t = \psi(t, S_t),$$

where η and ψ are continuous functions and continuously differentiable in the first component and twice continuously differentiable in the second. $\diamondsuit$

We observe that the value of Markovian strategy θ depends solely on time t and the price of asset S_t. We define function $V : [0, T] \times (0, \infty) \to (0, \infty)$ so that

$$\begin{aligned} V_t^\theta &= \eta_t B_t + \psi_t S_t && (3.6.9)\\ &= \eta(t, S_t)e^{rt} + \psi(t, S_t)S_t && (3.6.10)\\ &=: V(t, S_t). && (3.6.11)\end{aligned}$$

Now V satisfies condition of Itô's lemma (Corollary 3.2.46) and thus

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial s}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(t, S_t)d\langle S\rangle_t \tag{3.6.12}$$

We use equation (3.6.7) and recall that $d\langle S\rangle_t = \sigma^2 S_t^2 dt$ to get

$$\begin{aligned} dV(t, S_t) = &\left(\frac{\partial V}{\partial t}(t, S_t) + r\frac{\partial V}{\partial s}(t, S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial s^2}(t, S_t)\right)dt \\ &+ \sigma S_t \frac{\partial V}{\partial s}(t, S_t)dW_t^{\mathbb{Q}} \end{aligned} \tag{3.6.13}$$

Let us then define discounted value process $\widetilde{V}$ by $\widetilde{V}(t, S_t) = V(t, S_t)/B_t$ for all $t \in \mathbb{T}$. Integration by parts formula (Corollary 3.2.44) and equation (3.6.13) yield

$$\begin{aligned} d\widetilde{V}(t, S_t) &= B_t^{-1}dV(t, S_t) + V(t, S_t)d(B_t^{-1}) \\ &= \left(\frac{\partial V}{\partial t}(t, S_t) + r\frac{\partial V}{\partial s}(t, S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial s^2}(t, S_t) - rV(t, S_t)\right)B_t^{-1}dt \\ &+ \sigma \widetilde{S}_t \frac{\partial V}{\partial s}(t, S_t)dW_t^{\mathbb{Q}}, \end{aligned} \tag{3.6.14}$$

since $d(B_t^{-1}) = -rB_t^{-1}dt$. On the other hand, Proposition 3.5.7 and equation (3.6.5) implies, that a Markovian strategy is self-financing in Black-Scholes model if and only if

$$d\widetilde{V}(t, S_t) = \psi_t d\widetilde{S}_t = \sigma \widetilde{S}_t \psi_t dW_t^{\mathbb{Q}}. \tag{3.6.15}$$

These observations give us two results, when we compare equations (3.6.14) and (3.6.15). Let us first compare drift terms in aforementioned equations. These must coincide almost surely so

$$\frac{\partial V}{\partial t}(t, S_t) + r\frac{\partial V}{\partial s}(t, S_t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial s^2}(t, S_t) - rV(t, S_t) = 0, \tag{3.6.16}$$

since (3.6.5) has null drift. Equation (3.6.16) is called *Black-Scholes partial differential equation*. These type of partial differential equations are common in mathematical finance. Sometimes we cannot solve these equations analytically and have to resort to numerical methods, such as *finite difference method*. However, Black-Scholes partial differential equation can be solved various ways. Actually, we almost have a solution for it given by Feynman-Kac formula (Proposition 3.4.11) but the solution is for deterministic version. We omit this measure-theoretic matter (see Proposition 7.7 in [15] for details) and sum up our findings.

Theorem 3.6.17. *(Black-Scholes partial differential equation) Let us assume that θ is Markovian strategy with value given by function $V(t, S_t) := V_t^\theta$. Then strategy θ is self-financing if and only if V is a solution to partial differential equation*

$$\frac{\partial V}{\partial t}(t,s) + r\frac{\partial V}{\partial s}(t,s) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s) - rV(t,s) = 0. \tag{3.6.18}$$

It is noteworthy that equation above does not depend on μ. This fact has again roots in risk-neutrality. We move on to another implication of equations (3.6.14) and (3.6.15) which comes, when we compare diffusion terms. We observe that, in order to meet the self-financing condition, our strategy must satisfy

$$\psi_t = \frac{\partial V}{\partial s}(t, S_t) \quad \text{and} \quad \eta_t = e^{-rt}(V(t,S_t) - S_t\frac{\partial V}{\partial s}(t,S_t)). \tag{3.6.19}$$

The latter follows simply from (3.6.10).

3.6.3 Hedging and pricing

In this subsection, we restrict ourselves to European derivatives $X_T = f(S_T)$ with function f that is bounded from below, locally integrable on $(0, \infty)$ (this means that, it has finite integral for every compact subset of $(0, \infty)$), and such that there exists positive constants a and b so that

$$f(s) \leq ae^{a|\ln(s)|^{1+b}} \quad \text{for all } s \in (0, \infty). \tag{3.6.20}$$

From the practical financial aspect, these restrictions for f do not rule out common European derivatives that be found from the market but it is important for the existence of solutions to so-called *Cauchy problem* for the *heat equation*. We will not go deep into but mention, though, that Black-Scholes partial differential equation (see Theorem 3.6.17) can be converted to well-known heat equation with a convenient change of variables. For the detailed information, see Proposition 7.9 and Section 7.3 and Chapter 6 in [15].

Also some restrictions for the strategies is needed. We demand that there is a limit for debt one can go into. In this context, a strategy is called *admissible* if the value of that strategy is bounded below almost surely at any given time. Next we redefine replicability and an arbitrage.

Definition 3.6.21. We say that a European derivative X is *replicable* (or *hedgeable*), if there exists admissible self-financing Markovian strategy θ such that

$$V_T^\theta = X_T := f(S_T) \tag{3.6.22}$$

We call such θ a *replicating strategy* for X. Furthermore, a market model is called *compelete* if every European derivative, that has payoff given by function f, which satisfies (3.6.20) and prior conditions, is replicable. $\diamondsuit$

By admissibility, we rule out so-called doubling-strategies from the definition of an arbitrage. Without this condition, one can create strategies that have positive profit without initial capital or risk of losing money in the Black-Scholes model.

Definition 3.6.23. An admissible self-financing Markovian strategy θ is an *arbitrage* if $V_0^\theta = 0$ almost surely and there exist $u \in \mathbb{T} \setminus \{0\}$ such that

$$\mathbb{P}(V_u^\theta \geq 0) = 1 \quad \text{and} \quad \mathbb{P}((V_u^\theta > 0) > 0. \tag{3.6.24}$$

We say that market model is *arbitrage-free* if there does not exist an arbitrage. $\diamondsuit$

Theorem 3.6.25. *The Black-Scholes market model is both complete (with respect to Definition 3.6.21) and arbitrage-free (with respect to Definition 3.6.23).*

Proof. See Theorem 7.13 and Proposition 7.17 in [15]. $\square$

So if we assume that X is a European derivative with payoff $X_T = f(S_T)$ and f that satisfies conditions from the beginning of the subsection, then X is replicable. Furthermore, if strategy $\theta = \{(\eta_t, \psi_t) \,|\, t \in \mathbb{T}\}$, with value $V(t, S_t)$, is the replicating strategy for X, then θ is uniquely determined by (3.6.19), that is,

$$\psi_t = \frac{\partial V}{\partial s}(t, S_t) \quad \text{and} \quad \eta_t = e^{-rt}(V(t, S_t) - S_t \frac{\partial V}{\partial s}(t, S_t)). \tag{3.6.26}$$

Notice, that here V satisfies Black-Scholes partial differential equation (equation (3.6.18) in Theorem 3.6.17) with boundary condition $V(T, s) = f(s)$ given by replication demand. We can now use Feynman-Kac formula (equation (3.4.13) in Proposition 3.4.11) to get representation

$$V(t, s) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)} f(S_T) | S_t = s) \tag{3.6.27}$$

This must be the arbitrage-free price for derivative X, since strategy θ is replicating X. More closely, this formula gives time $t \in \mathbb{T}$ price of X provided that we know the value of the risky asset S_t. If we do not know, then price is a random variable depending on S_t. We can continue by manipulating solution to (3.6.7) as follows:

$$S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma W_T^{\mathbb{Q}}} = \underbrace{S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t^{\mathbb{Q}}}}_{=S_t} e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}})} =: S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}\, Z}, \tag{3.6.28}$$

where $Z = \frac{W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}}{\sqrt{T-t}}$ is independent of S_t and follows standard normal distribution $N(0,1)$ under $\mathbb{Q}$. Now we can rewrite (3.6.27) as

$$\begin{aligned} V(t,s) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(f(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,Z})|S_t = s) \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(f(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,Z})) \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty} f(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,x})\phi(x)dx, \end{aligned} \tag{3.6.29}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the probability density function of standard normal distribution. Generally, we cannot go any further than this without knowing the payout function f. The next important theorem gives a price in the case of European call option C which has payout given by $C_T = f(S_T) = (S_T - K)^+$, where K is the strike price. This is the celebrated Black-Scholes formula.

Theorem 3.6.30. *(Black-Scholes formula) The arbitrage-free price of a European call option $C(t, S_t)$ in the Black-Scholes model is given by*

$$C(t, S_t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2), \tag{3.6.31}$$

where the parameters are

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3.6.32}$$

with a relation $d_1 = d_2 + \sigma\sqrt{T-t}$, and Φ is the cumulative distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy. \tag{3.6.33}$$

Proof. Let us continue from (3.6.29) by putting $f(a) = (a-K)^+$ into it

$$C(t,s) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,x} - K)^+ e^{-\frac{x^2}{2}}dx \tag{3.6.34}$$

The integrand is non-zero, when

$$se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,x} - K > 0 \quad \iff \quad x > \frac{\ln(\frac{K}{s}) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} =: -d_2. \tag{3.6.35}$$

When this is satisfied, we can split the integral into two integrals so that

$$\begin{aligned} C(t,s) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,x} - K)e^{-\frac{x^2}{2}}dx \\ &= \frac{s}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T-t})^2}{2}}dx - \frac{e^{-r(T-t)}K}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{x^2}{2}}dx \end{aligned} \tag{3.6.36}$$

We continue by performing change of variables $y = -(x - \sigma\sqrt{T-t})$ in the first integral and $y = -x$ in the second. This yields

$$\begin{aligned}
C(t,s) &= \frac{s}{\sqrt{2\pi}} \int_{-\infty}^{d_2+\sigma\sqrt{T-t}} e^{-\frac{y^2}{2}} dy - \frac{e^{-r(T-t)}K}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}} dy \\
&= s\Phi(\underbrace{d_2 + \sigma\sqrt{T-t}}_{=d_1}) - e^{-r(T-t)} K\Phi(d_2),
\end{aligned} \tag{3.6.37}$$

which is what we strived for. □

3.6.4 Beyond Black-Scholes

The Black-Scholes model is simple and the formula has great benefits in practice, for example, it is easy to calculate and we can use it to calculate the *implied volatility*, if we know the price of European call option. But the model is by no means perfect. For example, it does not take into account transaction fees or dividends and it underestimates extreme movements in stock price that could occur during market crashes.

The Black-Scholes model also assumes that the volatility of stock σ is deterministic and constant with respect to time, which is not true in reality. This limitation can be overcome by defining a *stochastic volatility model* such as the *Heston model* that has dynamics

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \tag{3.6.38}$$
$$d\nu_t = \kappa(\rho - \nu_t)dt + \xi\sqrt{\nu_t} dW_t^\nu, \tag{3.6.39}$$

where μ, κ, ρ, ξ are constant parameters. Here the latter describes dynamics of the mean-reverting *Cox-Ingersoll-Ross* process which reverts to ρ and it can be also used to model *short rates*. As a short rate model it is however a bit out-of-date, since it can not reach negative value while interest rates in reality can.

The Black-Scholes model can not be used to price *bond options* or *swaptions*. For these pricing problems we need completely new frameworks and slightly more advanced techniques. For example, in the case of a European call option on *zero-coupon bond* we can use the *Vasicek model*

$$dr_t = \alpha(\mu - r_t)dt + \sigma dW_t, \tag{3.6.40}$$

as the underlying short rate model and use the *forward measure* to get a Black-Scholes type of formula for the option price. We will not present this though. See paper [11] by Jamshidian (1989). Yet another topic we did not cover comes from *Malliavin calculus* and can help us find hedging strategies for exotic options. Namely the *Clark-Ocone formula*, which states that every $\mathcal{F}_T$-measurable $\widetilde{F} \in \mathbb{D}_{1,2}$ can be represented as

$$\widetilde{F} = \mathbb{E}_{\mathbb{Q}}(\widetilde{F}) + \int_0^T \mathbb{E}_{\mathbb{Q}}(D_t\widetilde{F}|\mathcal{F}_t) dW_t^{\mathbb{Q}}, \tag{3.6.41}$$

where D_t marks a *Malliavin derivative* at time t. See Section 2.4.3 in [3] or more comprehensively "An introduction to Malliavin calculus with applications to economics" (1997) by Bernt Øksendal.

The matters we reviewed in this subsection have enough contents to fill theses on their own. As we see, even though this thesis has come to an end, we have only seen a glimpse of the enormous field of mathematical finance and the journey is only at the beginning.

Bibliography

[1] Bass, Richard. (2011) *Stochastic Processes (Cambridge Series in Statistical and Probabilistic Mathematics)*. Cambridge University Press.

[2] Björk, Tomas. (2004) *Arbitrage Theory in Continuous Time*, 2nd edition. Oxford University Press.

[3] Bouchard, Bruno.; Chassagneux, Jean-Francois. (2016) *Fundamentals and Advanced Techniques in Derivatives Hedging*. Springer.

[4] Black, Fischer.; Scholes, Myron. (1973) *The Pricing of Options and Corporate Liabilities*. Journal of Political Economy. **81** (3): 637-654.

[5] Cox, John.; Ross, Stephen.; Rubinstein, Mark. (1979) *Option Pricing: A Simplified Approach*. Journal of Financial Economics. **7** (3): 229-263.

[6] Dana, Rose-Anne.; Jeanblanc, Monique. (2003) *Financial Markets in Continuous Time*. Springer-Verlag.

[7] Delbaen, Freddy.; Schachermayer, Walter. (2006) *The Mathematics of Arbitrage*. Springer-Verlag.

[8] Durrett, Rick. (2010) *Probability: Theory and Examples*, 4th edition. Cambridge University Press.

[9] Elliot, Robert.; Kopp, Ekkehard. (1999) *Mathematics of Financial Markets*. Springer-Verlag.

[10] Föllmer, Hans.; Schied, Alexander. (2016) *Stochastic Finance (An Introduction in Discrete Time)*, 4nd edition. Walter de Gruyter.

[11] Jamshidian, Farshid. (1989) *An exact bond option pricing formula*. Journal of Finance. **44** (1): 205-209.

[12] Jarrow, Robert. (2018) *Continuous-Time Asset Pricing Theory (A Martingale-Based Approach)*. Springer.

[13] Merton, Robert. (1973) *Theory of Rational Option Pricing*. The Bell Journal of Economics and Management Science. **4** (1): 141-183.

[14] Mishura, Yuliya. (2016) *Financial Mathematics*. ISTE Press.

[15] Pascucci, Andrea. (2011) *PDE and Martingale Methods in Option Pricing*. Springer-Verlag.

[16] Pascucci, Andrea.; Runggaldier, Wolfgang. (2012) *Financial Mathematics: Theory and Problems for Multi-period Models*. Springer-Verlag.

[17] Shreve, Steven. (2004) *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Springer.

[18] Shreve, Steven. (2004) *Stochastic Calculus for Finance II: Continuous-Time Models.* Springer.

[19] Williams, David. (1991) *Probability with Martingales.* Cambridge University Press.